June 2017

AP Calculus BC students:

I look forward to having you in class next year. I'm writing to let you know that we will be diving right into the curriculum on the first day of school (Tuesday, September 5th) and there will be a test on our first unit, Limits and Continuity, on Friday, September 8th. To help you prepare ahead of time, I've put together a packet for you to complete. In AP Calculus BC, we use two text books – *Calculus* (3rd Edition) by Howard Anton and *Calculus: Single Variable* (5th Edition) by Deborah Hughes-Hallett. I've copied the relevant sections from each book. It is in your best interest to read these sections in their entirety before attempting the problems to become familiar with the material and how each author presents the content. The following problems will be **due on Wednesday, September 6th**:

From Anton:

2.4 Limits (An Intuitive Introduction): p. 93-95 #1-14

2.5 Limits (Computational Techniques): p. 106-107 #3-54 (multiples of 3 only – omit 27)

2.7 Continuity: p. 127 #1-18, 22

From Hughes-Hallett:

From 1.7 Introduction to Continuity: p. 49 #1-14

From 1.8 Limits: p. 57-58 #1-13

See you in September,

Ms. Hebert
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Sketch the graph of

(a) $f(x) = [x]$ (b) $f(x) = [x^2]$

(c) $f(x) = [x]^2$ (d) $f(x) = [\sin x]$.

39. A function f is called ***even*** if $f(-x) = f(x)$ for each x in the domain of f, and ***odd*** if $f(-x) = -f(x)$ for each such x. In each part, classify the function as even, odd, or neither.

(a) $f(x) = x^2$ (b) $f(x) = x^3$

(c) $f(x) = |x|$ (d) $f(x) = x + 1$

(e) $f(x) = \dfrac{x^5 - x}{1 + x^2}$ (f) $f(x) = 2$.

40. In Figure 2.3.1 we have sketched part of the graph of a function f (the part to the right of the y-axis). Complete the graph assuming

(a) f is an even function

(b) f is an odd function.

(See Exercise 39 for the definitions of even and odd functions.)

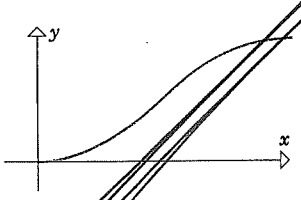


Figure 2.3.1

41. Classify the functions graphed in Figure 2.3.2 as even, odd, or neither. (See Exercise 39 for the definitions of even and odd.)

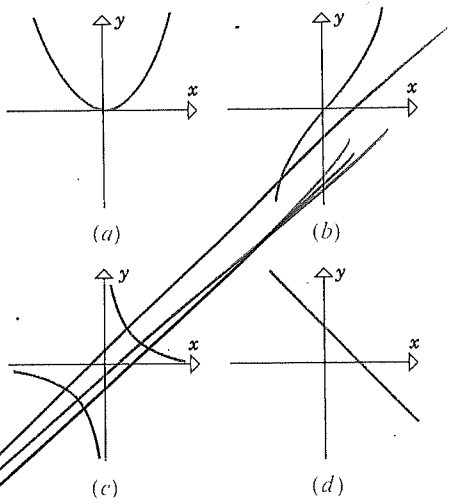


Figure 2.3.2

42. Can a function be both even and odd? (See Exercise 39 for the definitions of even and odd functions.)

43. Prove that the product of

(a) two even functions is an even function.

(b) two odd functions is an even function.

(c) an even and an odd function is an odd function.

(See Exercise 39 for the definitions of even and odd functions.)

44. Let a be a constant and suppose

$$f(a - x) = f(a + x)$$

for all x. What geometric property must the graph of f have?

ANTON:

2.4 LIMITS (AN INTUITIVE INTRODUCTION)

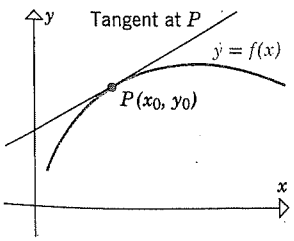


Figure 2.4.1

Calculus centers around the following two fundamental problems:

THE TANGENT PROBLEM. Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line tangent to the graph at P (Figure 2.4.1).

THE AREA PROBLEM. Given a function f, find the area between the graph of f and an interval $[a, b]$ on the x-axis (Figure 2.4.2).

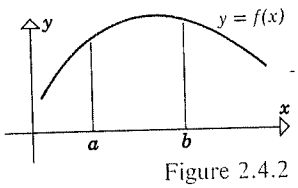


Figure 2.4.2

Traditionally, that portion of calculus arising from the tangent problem is called ***differential calculus*** and that arising from the area problem is called ***integral calculus***. As we shall see, however, the tangent and area problems are closely related, so that the distinction between differential calculus and integral calculus is often hard to discern.

In order to solve the tangent and area problems it is necessary to have a more precise understanding of the concepts of "tangent line" and "area." As we shall explain in this section, both of these concepts rest on a more fundamental concept, known as a "*limit*."

TANGENT LINES AND LIMITS

In plane geometry, a line is called ***tangent*** to a circle if it meets the circle at precisely one point (Figure 2.4.3*a*). However, this definition is not satisfactory for other kinds of curves. In Figure 2.4.3*b* the line meets the curve exactly once, yet is not a tangent, and in Figure 2.4.3*c* the line is tangent yet meets the curve more than once.

To define the concept of a tangent line so it applies to curves other than circles, we must view tangent lines another way. Consider a point P on a curve in the xy-plane. If Q is any point on the curve different from P, the line through P and Q is called a ***secant line*** for the curve (Figure 2.4.4). Intuition suggests that if we move the point Q along the curve toward P, the secant line will rotate toward a "limiting" position (Figure 2.4.5). The line T occupying this limiting position we consider to be the ***tangent line*** at P.

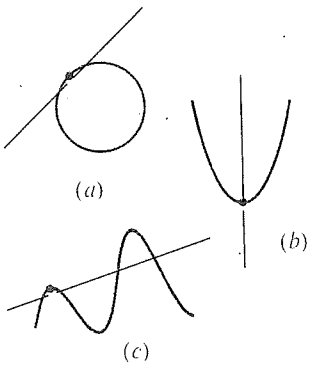


Figure 2.4.3

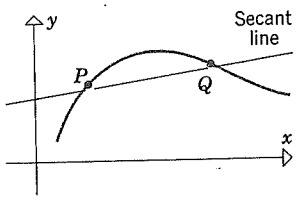


Figure 2.4.4

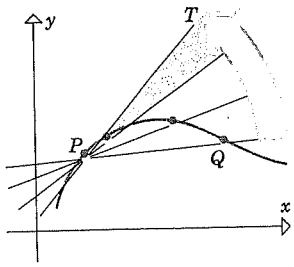


Figure 2.4.5

As suggested by Figure 2.4.6, this new concept of a tangent line coincides with the traditional concept, when applied to circles.

AREA AS A LIMIT

Just as the general notion of a tangent line leads to the concept of a "limit," so does the general notion of area. Areas of some plane regions can be calculated by subdividing them into a *finite* number of rectangles or triangles, then adding the areas of the constituent parts (Figure 2.4.7). However, for many regions a more general approach is needed. Consider the shaded region in Figure 2.4.8*a*. We can *approximate* the area of this region by inscribing rectangles of equal width under the curve and adding the areas of these rectangles (Figure 2.4.8*b*). Moreover, intuition suggests

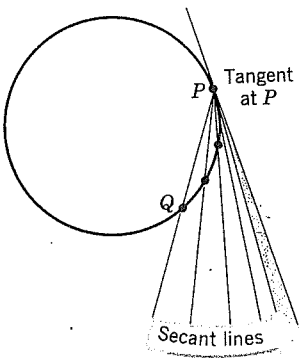
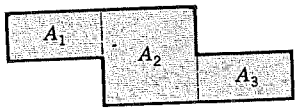
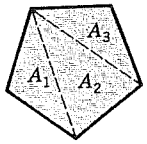


Figure 2.4.6



Area = $A_1 + A_2 + A_3$



Area = $A_1 + A_2 + A_3$

Figure 2.4.7

that if we repeat the process using more and more rectangles, then the rectangles will tend to fill in the gaps under the curve and our approximations will "approach" the exact area under the curve as a "limiting value" (Figure 2.4.8*c*).

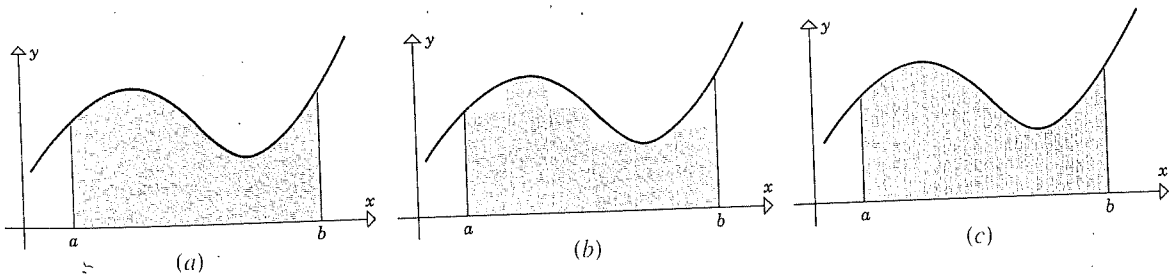


Figure 2.4.8

LIMITS In the foregoing discussion, we saw that the concepts of a tangent line and the area of a plane region ultimately rest on the notion of a "limit." In this section and the next few we shall investigate the notion of limit in more detail. Our development of limits in this text proceeds in three stages:

- First we discuss limits intuitively.
- Then we discuss methods for computing limits.
- Finally, we give a precise mathematical discussion of limits.

Limits can be used to describe how a function behaves as the independent variable moves toward a certain value. For example, consider the function

$$f(x) = \frac{\sin x}{x}$$

where x is in radians. Although this function is not defined at $x = 0$, it still makes sense to ask what happens to the values of $f(x)$ as x moves along the x-axis toward $x = 0$. To answer this question, we used a calculator set in the radian mode to obtain values of $f(x)$ at a succession of points moving along the *positive* x-axis toward $x = 0$. The results, which appear in Table 2.4.1, suggest that the values of $f(x)$ approach 1. We call the number 1 the ***limit*** of

$$\frac{\sin x}{x}$$

as x approaches 0 from the right side, and we write

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1 \tag{1}$$

In this expression, "lim" tells us that we are computing a limit; the symbol $x \to 0$ tells us that we are letting x approach 0; and the label "+" on $x \to 0$ tells us that x is approaching zero from the *right* side. We can also ask what happens to the values of

$$f(x) = \frac{\sin x}{x}$$

as x approaches 0 from the left side. From Table 2.4.2 [or from the fact that $f(-x) = f(x)$] it is evident that the values of $f(x)$ again approach 1. We denote this by writing

$$\lim_{x \to 0^-} \frac{\sin x}{x} = 1 \tag{2}$$

In (2), the label "$-$" on $x \to 0$ tells us that we are computing the limit as x approaches zero from the *left* side.

Table 2.4.1

x (RADIANS)	$\frac{\sin x}{x}$
1.0	0.84147
0.9	0.87036
0.8	0.89670
0.7	0.92031
0.6	0.94107
0.5	0.95885
0.4	0.97355
0.3	0.98507
0.2	0.99335
0.1	0.99833
0.01	0.99998

Table 2.4.2

x (RADIANS)	$\frac{\sin x}{x}$
-1.0	0.84147
-0.9	0.87036
-0.8	0.89670
-0.7	0.92031
-0.6	0.94107
-0.5	0.95885
-0.4	0.97355
-0.3	0.98507
-0.2	0.99335
-0.1	0.99833
-0.01	0.99998

REMARK. It is important to keep in mind that the limits given in (1) and (2) are really just guesses based on numerical evidence. It is conceivable that if we continue the calculator computations in Tables 2.4.1 and 2.4.2 for values of x still closer to 0, the pattern of values for $(\sin x)/x$ might change and approach some value different from 1 or perhaps approach no number at all. Moreover, no matter how much we enlarge our tables, this problem will persist since we can never be certain what happens for values of x not included in the table. To be certain that the limits in (1) and (2) are really correct, we need mathematical proof, and to give a mathematical proof, we need a precise mathematical definition of a limit.

These questions will be taken up later. However, in this section we shall rely completely on our intuition to obtain limits. Our main objective at this time is to introduce the mathematical notation associated with limits and to develop our intuitive understanding of limits geometrically.

2.4.1 NOTATION. If the value of $f(x)$ approaches the number L_1 as x approaches x_0 from the right side, we write

$$\lim_{x \to x_0^+} f(x) = L_1 \tag{3}$$

which is read, "the limit of $f(x)$ as x approaches x_0 from the right is equal to L_1."

2.4.2 NOTATION. If the value of $f(x)$ approaches the number L_2 as x approaches x_0 from the left side, we write

$$\lim_{x \to x_0^-} f(x) = L_2 \tag{4}$$

which is read, "the limit of $f(x)$ as x approaches x_0 from the left is equal to L_2."

2.4.3 NOTATION. If the limit from the left side is the same as the limit from the right side, that is,

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L$$

then we write

$$\lim_{x \to x_0} f(x) = L \tag{5}$$

which is read, "the limit of $f(x)$ as x approaches x_0 is equal to L."

The expressions

$$\lim_{x \to x_0^-} f(x) \quad \text{and} \quad \lim_{x \to x_0^+} f(x)$$

are called the ***one-sided limits of $f(x)$ at x_0*** and the expression

$$\lim_{x \to x_0} f(x)$$

is called the ***two-sided limit of $f(x)$ at x_0.***

Example 1 From (1) and (2)

$$\lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{\sin x}{x} = 1$$

Therefore,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \blacktriangleleft$$

The following examples illustrate limits geometrically.

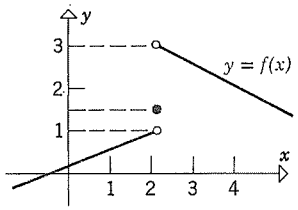


Figure 2.4.9

Example 2 Let f be the function whose graph is shown in Figure 2.4.9. As x approaches 2 from the left, $f(x)$ approaches 1, so

$$\lim_{x\to 2^-} f(x) = 1$$

As x approaches 2 from the right, $f(x)$ approaches 3, so that

$$\lim_{x\to 2^+} f(x) = 3$$

Since $f(2) = 1.5$, this example shows that the value of a function at a point, and the left- and right-hand limits there can all be different. ◀

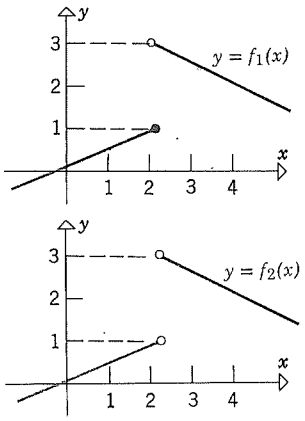


Figure 2.4.10

Example 3 The functions f_1 and f_2 whose graphs are shown in Figure 2.4.10 are identical to the function f in Example 2, except at the point $x = 2$; there we have $f(2) = 1.5$ and $f_1(2) = 1$, while $f_2(2)$ is undefined. However, even though the functions f, f_1, and f_2 behave differently *at* $x = 2$, their limits as x approaches 2 are the same, that is,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} f_1(x) = \lim_{x\to 2^-} f_2(x) = 1$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} f_1(x) = \lim_{x\to 2^+} f_2(x) = 3 \quad \blacktriangleleft$$

The foregoing example illustrates a general principle about limits that can be stated loosely as follows:

The limit of a function as the independent variable approaches a point does not depend on the value of the function *at* the point.

Thus, if we alter the value of a function f only at $x = x_0$, then we do not affect

$$\lim_{x\to x_0^-} f(x), \quad \lim_{x\to x_0^+} f(x), \quad \text{or} \quad \lim_{x\to x_0} f(x)$$

Function f_2 in Example 3 illustrates another idea: namely, that the left- and right-hand limits can have values at a point where the function itself is undefined.

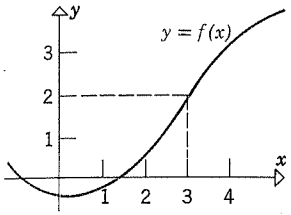


Figure 2.4.11

Example 4 Let f be the function whose graph is shown in Figure 2.4.11. From the graph we see that

$$\lim_{x \to 3^-} f(x) = 2 \quad \text{and} \quad \lim_{x \to 3^+} f(x) = 2$$

so that

$$\lim_{x \to 3} f(x) = 2$$

The graph also shows that $f(3) = 2$. Thus, the value of a function at a point and the one-sided and two-sided limits as x approaches that point may be equal. ◀

There is, in general, no guarantee that a function $f(x)$ actually has a limit as $x \to x_0^+$, $x \to x_0^-$, or $x \to x_0$. If there is no limit, then we say that ***the limit does not exist.*** Here are some examples.

Example 5 Let f be the function whose graph is shown in Figure 2.4.12. This graph is intended to convey the idea that as x approaches 6 from the right side, $f(x)$ oscillates between -3 and 3 infinitely often and with increasing frequency. Since $f(x)$ approaches no fixed value as x approaches 6 from the right, we say that

$$\lim_{x \to 6^+} f(x) \textit{ does not exist}$$

On the other hand, as x approaches 6 from the left side, $f(x)$ approaches 1, so that

$$\lim_{x \to 6^-} f(x) = 1 \quad ◀$$

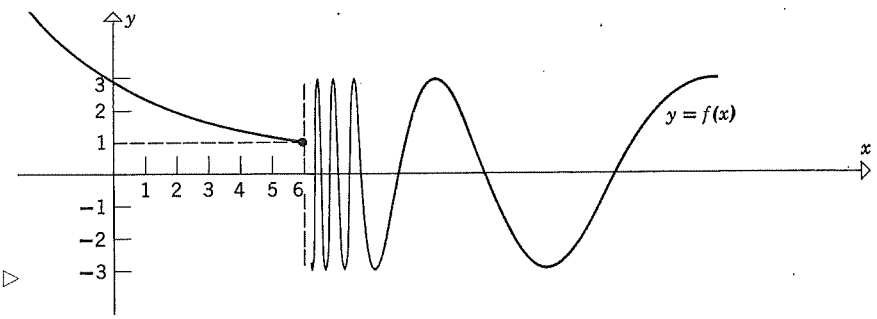


Figure 2.4.12 ▷

REMARK. Recall from 2.4.3 that in order to write $\lim_{x \to x_0} f(x) = L$ we must have $\lim_{x \to x_0^+} f(x) = L$ and $\lim_{x \to x_0^-} f(x) = L$. Thus, if either of the two one-sided limits fails to exist, then the two-sided limit fails to exist. Thus, in the foregoing example $\lim_{x \to 6} f(x)$ does not exist because $\lim_{x \to 6^+} f(x)$ fails to exist.

INFINITE LIMITS

Example 6 Let f be the function whose graph is shown in Figure 2.4.13. As x approaches 0 from the right side, $f(x)$ gets larger and larger without bound and consequently approaches no fixed finite value. Thus,

$$\lim_{x \to 0^+} f(x)$$

does not exist. In this case, we would write

$$\lim_{x \to 0^+} f(x) = +\infty \tag{6}$$

to indicate that the limit fails to exist because $f(x)$ is increasing without bound.

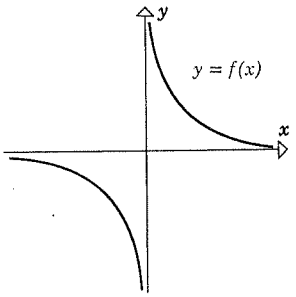


Figure 2.4.13

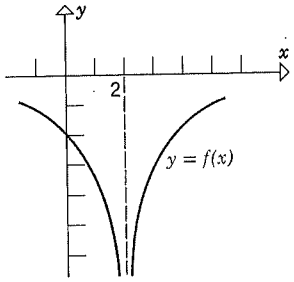
As x approaches 0 from the left side, $f(x)$ becomes more and more negative without bound and consequently approaches no fixed finite value. Thus,

$$\lim_{x \to 0^-} f(x)$$

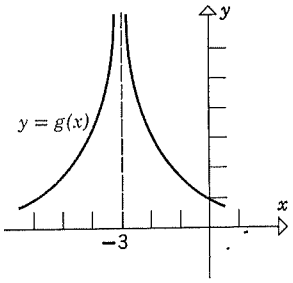
does not exist. In this case, we would write

$$\lim_{x \to 0^-} f(x) = -\infty \tag{7}$$

to indicate that the limit fails to exist because $f(x)$ is decreasing without bound. ◀



(a)



(b)

Figure 2.4.14

REMARK. It should be emphasized that the expressions $\lim_{x \to 0^+} f(x) = +\infty$ and $\lim_{x \to 0^-} f(x) = -\infty$ are simply descriptions of limits that do not exist. The symbols $+\infty$ and $-\infty$ do not represent numbers.

Example 7 Let f and g be the functions with graphs as shown in Figure 2.4.14. From the graph of f we see that

$$\lim_{x \to 2^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 2^+} f(x) = -\infty$$

We can indicate these two results more succinctly by writing

$$\lim_{x \to 2} f(x) = -\infty$$

From the graph of g we see that

$$\lim_{x \to -3^-} g(x) = +\infty \quad \text{and} \quad \lim_{x \to -3^+} g(x) = +\infty$$

More briefly, we shall write

$$\lim_{x \to -3} g(x) = +\infty \quad \blacktriangleleft$$

LIMITS AT INFINITY

So far we have used limits to describe how a function behaves as the independent variable approaches a fixed point on the x-axis. However, limits can also be used to describe how a function behaves as the independent variable moves "indefinitely far" from the origin along the x-axis. If x is allowed to increase without bound, then we write $x \to +\infty$ (read, "x approaches plus infinity"), and if x is allowed to decrease without bound then we write $x \to -\infty$ (read, "x approaches minus infinity").

Example 8 Let f be the function with the graph shown in Figure 2.4.15. As $x \to +\infty$ the graph of f approaches the line $y = 4$ so that the value of $f(x)$ approaches 4. We denote this by writing

$$\lim_{x \to +\infty} f(x) = 4$$

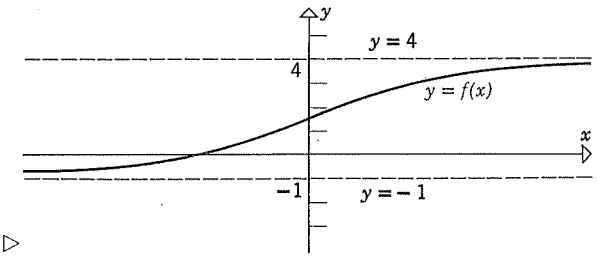


Figure 2.4.15 ▷

As $x \to -\infty$ the graph of f tends toward the line $y = -1$ and so the value of $f(x)$ approaches -1. We denote this by writing

$$\lim_{x \to -\infty} f(x) = -1 \quad \blacktriangleleft$$

Example 9 Let f be the function with the graph shown in Figure 2.4.16.

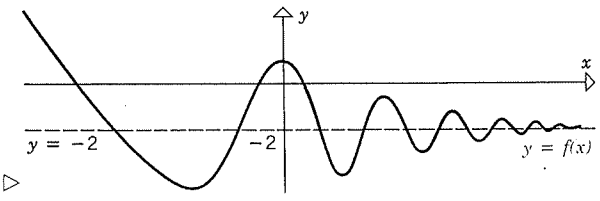


Figure 2.4.16 ▷

As x approaches $-\infty$, the value of $f(x)$ increases without bound and consequently approaches no fixed value. Thus,

$$\lim_{x \to -\infty} f(x)$$

does not exist. In this case, we would write

$$\lim_{x \to -\infty} f(x) = +\infty$$

to indicate that the limit fails to exist because $f(x)$ is increasing without bound.

As x approaches $+\infty$, the graph of f oscillates but tends toward the line $y = -2$. Thus, the value of $f(x)$ approaches -2 as x approaches $+\infty$ and we write

$$\lim_{x \to +\infty} f(x) = -2 \quad \blacktriangleleft$$

Example 10 Let f be the function with the graph shown in Figure 2.4.17. As x approaches $-\infty$, the value of $f(x)$ decreases without bound and consequently approaches no fixed finite value. Thus,

$$\lim_{x \to -\infty} f(x)$$

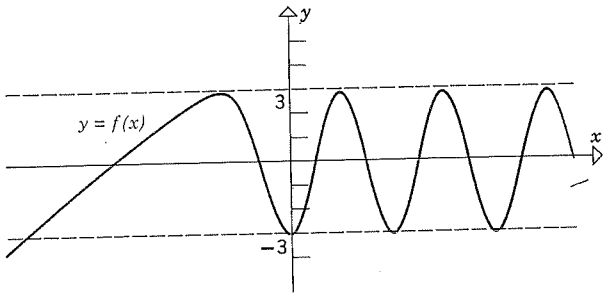


Figure 2.4.17

does not exist. In this case, we would write

$$\lim_{x \to -\infty} f(x) = -\infty$$

to indicate that the limit fails to exist because $f(x)$ is decreasing without bound.

As x approaches $+\infty$, the value of $f(x)$ oscillates between -3 and $+3$ and consequently approaches no fixed value. Thus,

$$\lim_{x \to +\infty} f(x)$$

does not exist. There is no special notation used to describe limits that fail to exist because of oscillation. $\blacktriangleleft$

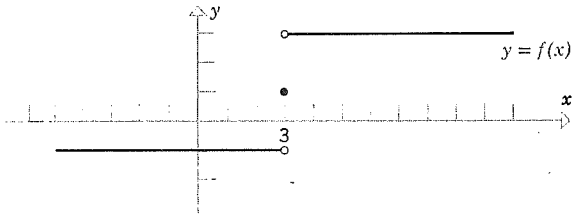
▶ Exercise Set 2.4

1. For the function f graphed below, find

(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$ (d) $f(3)$

(e) $\lim_{x\to -\infty} f(x)$ (f) $\lim_{x\to +\infty} f(x)$.

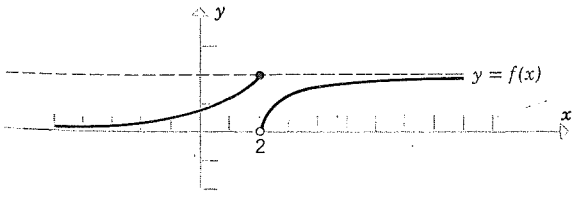


2. For the function f graphed below, find

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to 2} f(x)$ (d) $f(2)$

(e) $\lim_{x\to -\infty} f(x)$ (f) $\lim_{x\to +\infty} f(x)$.

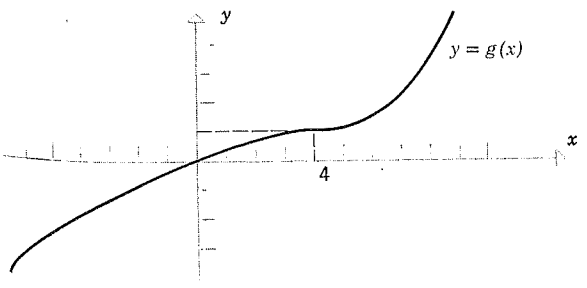


3. For the function g graphed below, find

(a) $\lim_{x\to 4^-} g(x)$ (b) $\lim_{x\to 4^+} g(x)$

(c) $\lim_{x\to 4} g(x)$ (d) $g(4)$

(e) $\lim_{x\to -\infty} g(x)$ (f) $\lim_{x\to +\infty} g(x)$.

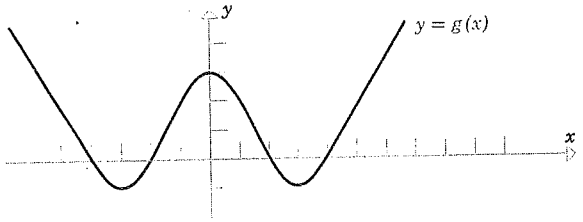


4. For the function g graphed below, find

(a) $\lim_{x\to 0^-} g(x)$ (b) $\lim_{x\to 0^+} g(x)$

(c) $\lim_{x\to 0} g(x)$ (d) $g(0)$

(e) $\lim_{x\to -\infty} g(x)$ (f) $\lim_{x\to +\infty} g(x)$.

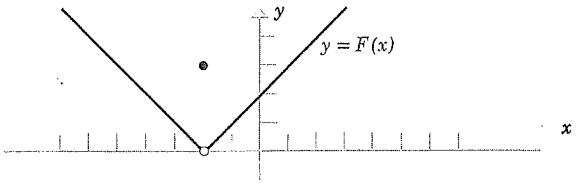


5. For the function F graphed below, find

(a) $\lim_{x\to -2^-} F(x)$ (b) $\lim_{x\to -2^+} F(x)$

(c) $\lim_{x\to -2} F(x)$ (d) $F(-2)$

(e) $\lim_{x\to -\infty} F(x)$ (f) $\lim_{x\to +\infty} F(x)$.

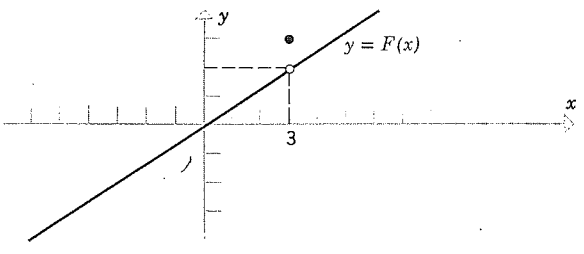


6. For the function F graphed below, find

(a) $\lim_{x\to 3^-} F(x)$ (b) $\lim_{x\to 3^+} F(x)$

(c) $\lim_{x\to 3} F(x)$ (d) $F(3)$

(e) $\lim_{x\to -\infty} F(x)$ (f) $\lim_{x\to +\infty} F(x)$.

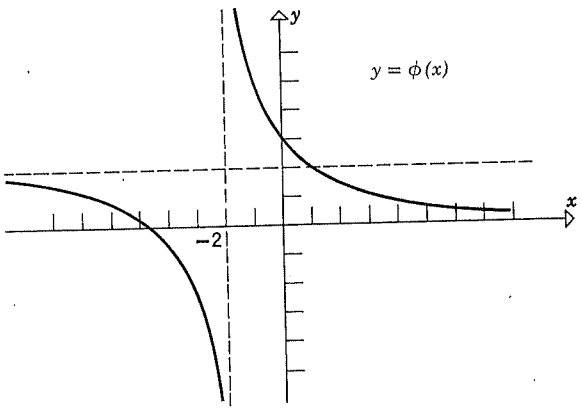


7. For the function ϕ graphed below, find

(a) $\lim_{x \to -2^-} \phi(x)$ (b) $\lim_{x \to -2^+} \phi(x)$

(c) $\lim_{x \to -2} \phi(x)$ (d) $\phi(-2)$

(e) $\lim_{x \to -\infty} \phi(x)$ (f) $\lim_{x \to +\infty} \phi(x)$.

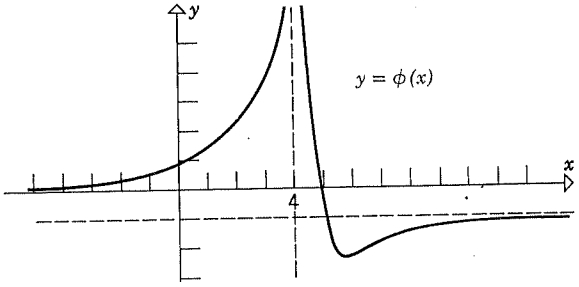


8. For the function ϕ graphed below, find

(a) $\lim_{x \to 4^-} \phi(x)$ (b) $\lim_{x \to 4^+} \phi(x)$

(c) $\lim_{x \to 4} \phi(x)$ (d) $\phi(4)$

(e) $\lim_{x \to -\infty} \phi(x)$ (f) $\lim_{x \to +\infty} \phi(x)$.

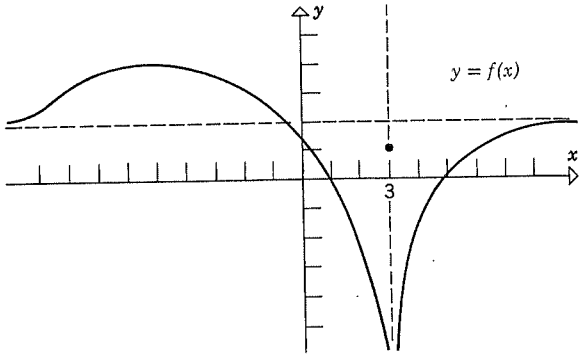


9. For the function f in the following graph, find

(a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$

(c) $\lim_{x \to 3} f(x)$ (d) $f(3)$

(e) $\lim_{x \to -\infty} f(x)$ (f) $\lim_{x \to +\infty} f(x)$.

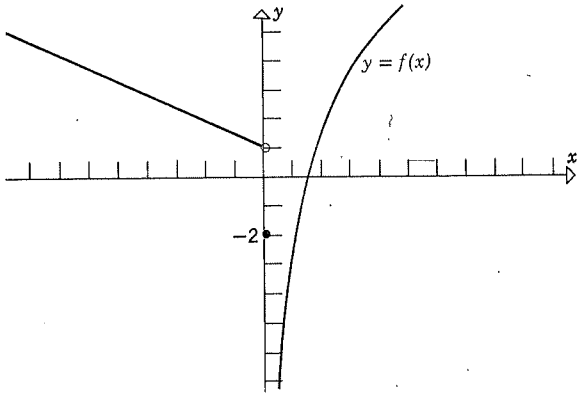


10. For the function f graphed below, find

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$

(c) $\lim_{x \to 0} f(x)$ (d) $f(0)$

(e) $\lim_{x \to -\infty} f(x)$ (f) $\lim_{x \to +\infty} f(x)$.

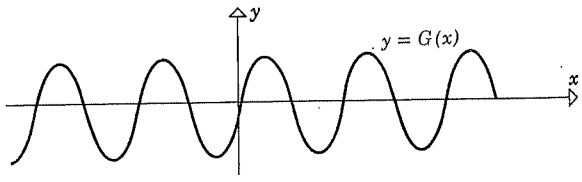


11. For the function G graphed below, find

(a) $\lim_{x \to 0^-} G(x)$ (b) $\lim_{x \to 0^+} G(x)$

(c) $\lim_{x \to 0} G(x)$ (d) $G(0)$

(e) $\lim_{x \to -\infty} G(x)$ (f) $\lim_{x \to +\infty} G(x)$.

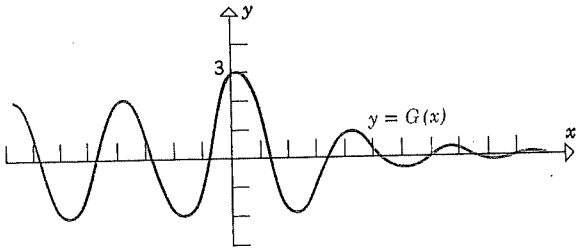


12. For the function G graphed below, find

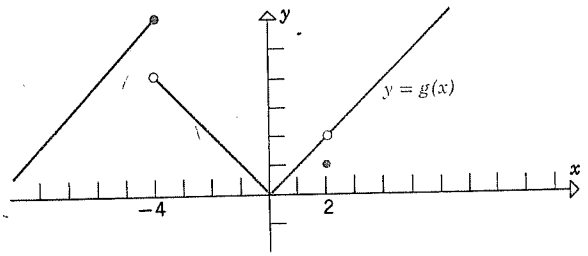
(a) $\lim_{x \to 0^-} G(x)$ (b) $\lim_{x \to 0^+} G(x)$

(c) $\lim_{x \to 0} G(x)$ (d) $G(0)$

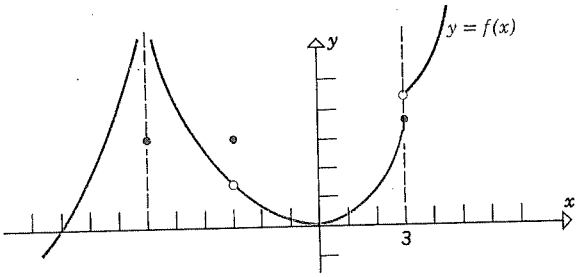
(e) $\lim_{x \to -\infty} G(x)$ (f) $\lim_{x \to +\infty} G(x)$.



13. Consider the function g in the following graph. For what values of x_0 does $\lim_{x \to x_0} g(x)$ exist?



14. Consider the function f graphed below. For what values of x_0 does $\lim_{x \to x_0} f(x)$ exist?



2.5 LIMITS (COMPUTATIONAL TECHNIQUES)

In the last section we concentrated on the graphical interpretation of limits. In this section we shall discuss techniques for finding limits directly from the formula for a function. Again our results will be based on intuition. Mathematical proofs of the results here will have to wait until we have a precise mathematical definition of a limit.

The limits discussed in our first four examples are the basic tools used to solve more complicated limit problems.

Example 1 Since a constant function $f(x) = k$ has the same value k everywhere, it follows that at each point a

$$\lim_{x \to a} k = k$$

and also

$$\lim_{x \to +\infty} k = \lim_{x \to -\infty} k = k$$

(Figure 2.5.1). These results can be summarized by the statement:

The limit of a constant function k at any point is k. ◀

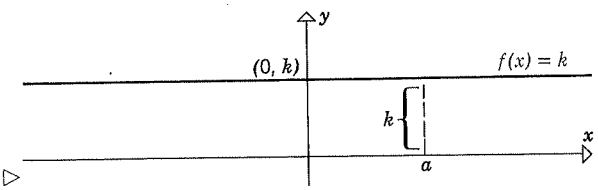


Figure 2.5.1 ▷

Example 2 The limit

$$\lim_{x \to a} x = a$$

is self-evident. For example,

$$\lim_{x \to 5} x = 5, \quad \lim_{x \to 0} x = 0, \quad \lim_{x \to -2} x = -2$$ ◀

Example 3 Find the limits

$$\lim_{x \to 0^+} \frac{1}{x}, \quad \lim_{x \to 0^-} \frac{1}{x}, \quad \lim_{x \to 0} \frac{1}{x}$$

Solution. It will be evident from some simple calculations that none of these limits exist. As x approaches 0 from the right, the value of $1/x$ gets larger and larger without bound. For example, if x successively assumes values

$$x = 1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10{,}000}, \ldots$$

then $1/x$ successively assumes values

$$\frac{1}{x} = 1, 10, 100, 1000, 10{,}000, \ldots$$

Thus,

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

As x approaches 0 from the left, the value of $1/x$ decreases (that is, becomes more and more negative) without bound. For example, if x successively assumes values

$$x = -1, -\frac{1}{10}, -\frac{1}{100}, -\frac{1}{1000}, -\frac{1}{10{,}000}, \ldots$$

then $1/x$ successively assumes values

$$\frac{1}{x} = -1, -10, -100, -1000, -10{,}000, \ldots$$

Thus,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

It follows that

$$\lim_{x \to 0} \frac{1}{x} \text{ does not exist}$$

since the one-sided limits as x approaches zero do not exist. Moreover, we cannot even write

$$\lim_{x \to 0} \frac{1}{x} = +\infty \quad \text{or} \quad \lim_{x \to 0} \frac{1}{x} = -\infty$$

(Why?)

The reader should check that the results in this example are consistent with the graph of $1/x$ (see Figure 1.3.20). ◀

Example 4 Find the limits

$$\lim_{x \to +\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x}$$

Solution. These limits are also evident from some sample calculations. To investigate the case $x \to +\infty$, assume x takes on successive values

$$x = 1, 10, 100, 1000, 10{,}000, \ldots$$

Then $1/x$ successively assumes values

$$\frac{1}{x} = 1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10{,}000}, \ldots$$

It is clear from these values of $1/x$ that

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

To investigate the case $x \to -\infty$, assume x takes on successive values

$$x = -1, -10, -100, -1000, -10{,}000, \ldots$$

Then $1/x$ successively assumes values

$$\frac{1}{x} = -1, -\frac{1}{10}, -\frac{1}{100}, -\frac{1}{1000}, -\frac{1}{10{,}000}, \ldots$$

It is clear from these values of $1/x$ that

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

The reader should check that the results in this example are consistent with the graph of $1/x$ (see Figure 1.3.20). ◀

The results in our first four examples, together with the fundamental properties of limits given in the following theorem, can be used to help solve more complicated limit problems. Parts of this theorem are proved in Appendix 2.

2.5.1 THEOREM. *Let* $\lim$ *stand for one of the limits* $\lim_{x \to a}$, $\lim_{x \to a^-}$, $\lim_{x \to a^+}$, $\lim_{x \to +\infty}$, *or* $\lim_{x \to -\infty}$. *If* $L_1 = \lim f(x)$ *and* $L_2 = \lim g(x)$ *both exist, then:*

(*a*) $\lim [f(x) + g(x)] = \lim f(x) + \lim g(x) = L_1 + L_2$.

(*b*) $\lim [f(x) - g(x)] = \lim f(x) - \lim g(x) = L_1 - L_2$.

(*c*) $\lim [f(x)g(x)] = \lim f(x) \lim g(x) = L_1 L_2$.

(*d*) $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)} = \dfrac{L_1}{L_2}$ *if* $L_2 \neq 0$.

(*e*) $\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)} = \sqrt[n]{L_1}$ *provided* $L_1 > 0$ *if* n *is even.*

In other words, this theorem states:

(*a*) *The limit of a sum is the sum of the limits.*
(*b*) *The limit of a difference is the difference of the limits.*
(*c*) *The limit of a product is the product of the limits.*
(*d*) *The limit of a quotient is the quotient of the limits provided the limit of the denominator is nonzero.*
(*e*) *The limit of an nth root is the nth root of the limit.*

REMARK. Although results (*a*) and (*c*) are stated for two functions f and g, these results hold as well for any finite number of functions; that is, if

$$\lim f_1(x), \lim f_2(x), \ldots, \lim f_n(x)$$

all exist, then

$$\lim [f_1(x) + f_2(x) + \cdots + f_n(x)] = \lim f_1(x) + \lim f_2(x) + \cdots + \lim f_n(x)$$

and

$$\lim [f_1(x)f_2(x) \cdots f_n(x)] = \lim f_1(x) \lim f_2(x) \cdots \lim f_n(x) \tag{1}$$

In particular, if $f_1, f_2, \ldots, f_n$ are all the same function f, then (1) reduces to

$$\lim [f(x)]^n = [\lim f(x)]^n \tag{2}$$

From (2) we obtain the useful result

$$\lim_{x \to a} x^n = [\lim_{x \to a} x]^n = a^n \tag{3}$$

For example,

$$\lim_{x \to 3} x^4 = 3^4 = 81$$

Another useful result follows from Example 1 and part (*c*) of Theorem 2.5.1. If k is a constant, then

$$\lim kf(x) = \lim k \lim f(x) = k \lim f(x) \tag{4}$$

The first and last expressions in (4) tell us:

A constant factor can be moved through a limit sign.

Example 5 Find $\lim_{x \to 5} (x^2 - 4x + 3)$ and justify each step.

Solution.

$$\begin{aligned}
\lim_{x \to 5} (x^2 - 4x + 3) &= \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3 && \text{[Theorem 2.5.1(}a\text{), (}b\text{)]} \\
&= \lim_{x \to 5} x^2 - 4 \lim_{x \to 5} x + \lim_{x \to 5} 3 && \text{[Equation (4)]} \\
&= 5^2 - 4(5) + 3 && \text{[Equation (3)]} \\
&= 8 \quad \blacktriangleleft
\end{aligned}$$

Example 6 Show that for any polynomial

$$p(x) = c_0 + c_1x + \cdots + c_nx^n$$

and any real number a,

$$\lim_{x\to a} p(x) = c_0 + c_1a + \cdots + c_na^n = p(a)$$

Solution.

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a} (c_0 + c_1x + \cdots + c_nx^n)\\ &= \lim_{x\to a} c_0 + \lim_{x\to a} c_1x + \cdots + \lim_{x\to a} c_nx^n\\ &= \lim_{x\to a} c_0 + c_1 \lim_{x\to a} x + \cdots + c_n \lim_{x\to a} x^n\\ &= c_0 + c_1a + \cdots + c_na^n = p(a)\end{aligned}$$ ◀

REMARK. Example 6 states that for a polynomial $p(x)$, the limit as x approaches a is equal to the value of the polynomial at a. If we apply this result to the limit problem in Example 5, we can bypass the intermediate steps and write immediately

$$\lim_{x\to 5} (x^2 - 4x + 3) = 5^2 - 4(5) + 3 = 8$$

Example 7 Find

$$\lim_{x\to 2} \frac{5x^3 + 4}{x - 3}$$

Solution. Using the result of Example 6 and part (*d*) of Theorem 2.5.1, we obtain

$$\lim_{x\to 2} \frac{5x^3 + 4}{x - 3} = \frac{\lim_{x\to 2} (5x^3 + 4)}{\lim_{x\to 2} (x - 3)} = \frac{5 \cdot 2^3 + 4}{2 - 3} = -44$$ ◀

Example 8 Find

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$$

Solution. The method used in the previous example cannot be used here since part (*d*) of Theorem 2.5.1 does not apply (the limit of the denominator is 0). However, by factoring the numerator and canceling, we obtain

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4 \quad \blacktriangleleft$$

Though correct, the second equality in this calculation needs some justification. We pointed out in Example 8 of Section 2.1 that to cancel the $x-2$ factors from

$$h(x)=\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{(x-2)}$$

we must write

$$h(x)=x+2, \quad x\neq 2$$

in order not to alter the domain of h. However, the functions

$$f(x)=x+2 \quad \text{and} \quad h(x)=x+2, \quad x\neq 2$$

differ only at $x=2$, so that their limits as x approaches 2 are the same. Thus,

$$\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)$$

even though

$$\frac{(x-2)(x+2)}{x-2} \quad \text{and} \quad x+2$$

are different functions.

REMARK. The technique in the foregoing example is extremely important. In any limit problem where the numerator and denominator both approach zero it is often possible to obtain the limit by performing some algebraic manipulations. Sometimes this will not work and more advanced techniques, to be studied later, are required to find the limit.

Example 9 Find

$$\lim_{x\to 4}\frac{2-x}{(x-4)(x+2)}$$

Solution. Again, part (*d*) of Theorem 2.5.1 cannot be applied since the limit of the denominator is zero. There are no common factors in the numerator and denominator as there were in Example 8, so we must use a different approach. This kind of problem is best attacked by looking at the one-sided limits.

First consider

$$\lim_{x\to 4^+} \frac{2-x}{(x-4)(x+2)}$$

As x approaches 4 from the right, the numerator is a negative quantity approaching -2 and the denominator a positive quantity approaching 0. Consequently the ratio is a negative quantity that decreases without bound. That is,

$$\lim_{x\to 4^+} \frac{2-x}{(x-4)(x+2)} = -\infty \tag{5}$$

Now consider

$$\lim_{x\to 4^-} \frac{2-x}{(x-4)(x+2)}$$

As x approaches 4 from the left, the numerator is eventually a negative quantity approaching -2 and the denominator is a negative quantity approaching 0. Consequently, the ratio is a positive quantity that increases without bound; that is,

$$\lim_{x\to 4^-} \frac{2-x}{(x-4)(x+2)} = +\infty \tag{6}$$

From (5) and (6)

$$\lim_{x\to 4} \frac{2-x}{(x-4)(x+2)}$$

does not exist. ◀

Next we shall consider methods for finding limits of rational functions as $x \to +\infty$ or $x \to -\infty$. But first we need some preliminary results.

In Example 4 we obtained the limits

$$\lim_{x\to+\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0$$

Consequently, for any positive integer n,

$$\lim_{x\to+\infty} \frac{1}{x^n} = \left(\lim_{x\to+\infty} \frac{1}{x}\right)^n = 0 \tag{7}$$

and

$$\lim_{x\to-\infty} \frac{1}{x^n} = \left(\lim_{x\to-\infty} \frac{1}{x}\right)^n = 0 \tag{8}$$

Results (7) and (8) are also evident from the graph of $1/x^n$, which has one of the two general shapes shown in Fig. 2.5.2, depending on whether n is even or odd.

If we divide numerator and denominator of a rational function by the highest power of x that occurs in the function, then all the powers of x become constants or powers of $1/x$. The next two examples show how this observation together with (7) and (8) can be used to find $\lim_{x\to+\infty}$ or $\lim_{x\to-\infty}$ for rational functions.

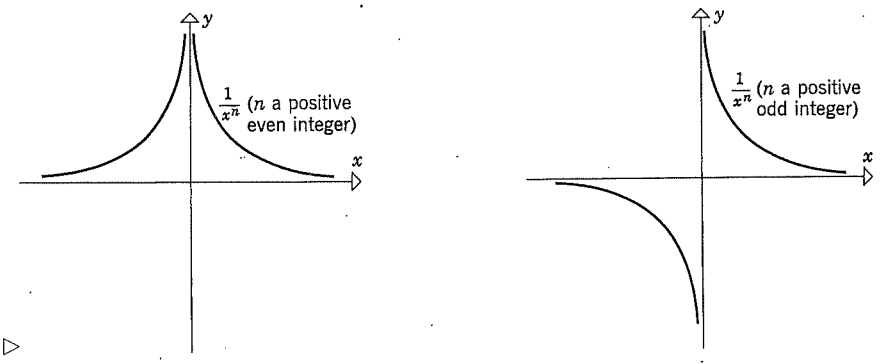


Figure 2.5.2 ▷

Example 10 Find

$$\lim_{x\to+\infty} \frac{3x+5}{6x-8}.$$

Solution. Divide the numerator and denominator by the highest power of x that occurs; this is $x^1 = x$. We obtain

$$\lim_{x\to+\infty} \frac{3x+5}{6x-8} = \lim_{x\to+\infty} \frac{3+5/x}{6-8/x} = \frac{\lim_{x\to+\infty}(3+5/x)}{\lim_{x\to+\infty}(6-8/x)}$$

$$= \frac{\lim_{x\to+\infty} 3 + \lim_{x\to+\infty} 5/x}{\lim_{x\to+\infty} 6 - \lim_{x\to+\infty} 8/x} = \frac{3+5\lim_{x\to+\infty} 1/x}{6-8\lim_{x\to+\infty} 1/x}$$

$$= \frac{3+(5\cdot 0)}{6-(8\cdot 0)} = \frac{1}{2} \quad \blacktriangleleft$$

Example 11 Find

$$\lim_{x\to-\infty} \frac{4x^2-x}{2x^3-5}$$

Solution. Divide numerator and denominator by the highest power of x that occurs, namely x^3. We obtain

$$\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}=\lim_{x\to-\infty}\frac{4/x-1/x^2}{2-5/x^3}=\frac{\lim\limits_{x\to-\infty}(4/x-1/x^2)}{\lim\limits_{x\to-\infty}(2-5/x^3)}$$

$$=\frac{\lim\limits_{x\to-\infty}4/x-\lim\limits_{x\to-\infty}1/x^2}{\lim\limits_{x\to-\infty}2-\lim\limits_{x\to-\infty}5/x^3}=\frac{4\lim\limits_{x\to-\infty}1/x-\lim\limits_{x\to-\infty}1/x^2}{2-5\lim\limits_{x\to-\infty}1/x^3}$$

$$=\frac{(4\cdot 0)-0}{2-(5\cdot 0)}=\frac{0}{2}=0 \quad \blacktriangleleft$$

Example 12 Find

$$\lim_{x\to+\infty}\sqrt[3]{\frac{3x+5}{6x-8}}$$

Solution.

$$\lim_{x\to+\infty}\sqrt[3]{\frac{3x+5}{6x-8}}=\sqrt[3]{\lim_{x\to+\infty}\frac{3x+5}{6x-8}}=\sqrt[3]{\frac{1}{2}} \quad \blacktriangleleft$$

Theorem 2.5.1, part (e) — Example 10

Example 13 Find

$$\lim_{x\to+\infty}\frac{3-2x^4}{x+1}$$

Solution. Dividing numerator and denominator by x^4, we obtain

$$\lim_{x\to+\infty}\frac{3-2x^4}{x+1}=\lim_{x\to+\infty}\frac{3/x^4-2}{1/x^3+1/x^4} \tag{9}$$

Since the limit of the denominator in the right side of (9) is zero, we cannot apply Theorem 2.5.1. However, we can argue as follows. In the numerator, $3/x^4$ approaches 0 as $x\to+\infty$ so that

$$\frac{3}{x^4}-2$$

is eventually negative and approaches -2. As $x\to+\infty$, the denominator is a positive quantity approaching 0. Consequently, as $x\to+\infty$, the ratio is eventually a negative quantity that decreases without bound. That is,

$$\lim_{x\to+\infty}\frac{3-2x^4}{x+1}=\lim_{x\to+\infty}\frac{3/x^4-2}{1/x^3+1/x^4}=-\infty \quad \blacktriangleleft$$

Example 14 Find

$$\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}$$

Solution. Since we are letting $x\to+\infty$, it would be helpful to manipulate the function so that the powers of x become powers of $1/x$. To achieve this, we divide numerator and denominator by x. In the numerator, we write x as

$$x=\sqrt{x^2}$$

This is justified because x eventually assumes positive values as $x\to+\infty$, so that

$$\sqrt{x^2}=|x|=x$$

Thus,

$$\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to+\infty}\frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/x}=\lim_{x\to+\infty}\frac{\sqrt{1+2/x^2}}{3-6/x}$$

$$=\frac{\lim\limits_{x\to+\infty}\sqrt{1+2/x^2}}{\lim\limits_{x\to+\infty}(3-6/x)}=\frac{\sqrt{\lim\limits_{x\to+\infty}(1+2/x^2)}}{\lim\limits_{x\to+\infty}(3-6/x)}$$

$$=\frac{\sqrt{\lim\limits_{x\to+\infty}1+2\lim\limits_{x\to+\infty}1/x^2}}{\lim\limits_{x\to+\infty}3-6\lim\limits_{x\to+\infty}1/x}=\frac{\sqrt{1+(2\cdot0)}}{3-(6\cdot0)}=\frac{1}{3} \quad \blacktriangleleft$$

Example 15 Find

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}$$

Solution. The function here is the same as in Example 14. However, in this problem we want $\lim\limits_{x\to-\infty}$ rather than $\lim\limits_{x\to+\infty}$. As in Example 14, we divide numerator and denominator by x, but in the numerator we write x as

$$x=-\sqrt{x^2}$$

This is justified because x eventually assumes negative values as $x\to-\infty$, so that

$$\sqrt{x^2} = |x| = -x$$

We obtain

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to-\infty} \frac{\sqrt{x^2+2}/(-\sqrt{x^2})}{3-(6/x)}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{1+2/x^2}}{3-(6/x)} = -\frac{1}{3} \quad \blacktriangleleft$$

Example 16 Find $\lim_{x\to3} f(x)$ for

$$f(x) = \begin{cases} x^2 - 5, & x \le 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$$

Solution. As x approaches 3 from the left, the formula for f is

$$f(x) = x^2 - 5$$

so that

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} (x^2 - 5) = 3^2 - 5 = 4$$

As x approaches 3 from the right, the formula for f is

$$f(x) = \sqrt{x+13}$$

so that

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} \sqrt{x+13} = \sqrt{\lim_{x\to3^+} (x+13)} = \sqrt{16} = 4$$

Therefore,

$$\lim_{x\to3} f(x) = 4$$

since the one-sided limits are equal. $\blacktriangleleft$

▶ Exercise Set 2.5

Find the limits in Exercises 1–50.

1. $\lim_{x\to8} 7$.
2. $\lim_{x\to-\infty} (-3)$.
3. $\lim_{x\to0^+} \pi$.
4. $\lim_{x\to-2} 3x$.
5. $\lim_{y\to3^+} 12y$.
6. $\lim_{h\to+\infty} (-2h)$.
7. $\lim_{x\to5} \sqrt{x^3 - 3x - 1}$.
8. $\lim_{x\to0^-} (x^4 + 12x^3 - 17x + 2)$.

9. $\lim_{y\to -1}(y^6 - 12y + 1)$.

10. $\lim_{x\to 3}\dfrac{x^2 - 2x}{x + 1}$.

11. $\lim_{y\to 2^-}\dfrac{(y-1)(y-2)}{y+1}$.

12. $\lim_{x\to 0}\dfrac{6x - 9}{x^3 - 12x + 3}$.

13. $\lim_{x\to 4}\dfrac{x^2 - 16}{x - 4}$.

14. $\lim_{t\to -2}\dfrac{t^3 + 8}{t + 2}$.

15. $\lim_{x\to 1^+}\dfrac{x^4 - 1}{x - 1}$.

16. $\lim_{x\to 2}\dfrac{x^2 - 4x + 4}{x^2 + x - 6}$.

17. $\lim_{x\to -1}\dfrac{x^2 + 6x + 5}{x^2 - 3x - 4}$.

18. $\lim_{t\to 1}\dfrac{t^3 + t^2 - 5t + 3}{t^3 - 3t + 2}$.

19. $\lim_{x\to +\infty}\dfrac{3x + 1}{2x - 5}$.

20. $\lim_{x\to +\infty}\dfrac{1}{x - 12}$.

21. $\lim_{y\to -\infty}\dfrac{3}{y + 4}$.

22. $\lim_{x\to +\infty}\dfrac{5x^2 + 7}{3x^2 - x}$.

23. $\lim_{x\to -\infty}\dfrac{x - 2}{x^2 + 2x + 1}$.

24. $\lim_{s\to +\infty}\sqrt[3]{\dfrac{3s^7 - 4s^5}{2s^7 + 1}}$.

25. $\lim_{x\to -\infty}\dfrac{\sqrt{5x^2 - 2}}{x + 3}$.

26. $\lim_{x\to +\infty}\dfrac{\sqrt{5x^2 - 2}}{x + 3}$.

27. $\lim_{y\to -\infty}\dfrac{2 - y}{\sqrt{7 + 6y^2}}$.

28. $\lim_{y\to +\infty}\dfrac{2 - y}{\sqrt{7 + 6y^2}}$.

29. $\lim_{x\to -\infty}\dfrac{\sqrt{3x^4 + x}}{x^2 - 8}$.

30. $\lim_{x\to +\infty}\dfrac{\sqrt{3x^4 + x}}{x^2 - 8}$.

31. $\lim_{x\to 3^+}\dfrac{x}{x - 3}$.

32. $\lim_{x\to 3^-}\dfrac{x}{x - 3}$.

33. $\lim_{x\to 3}\dfrac{x}{x - 3}$.

34. $\lim_{x\to 2^+}\dfrac{x}{x^2 - 4}$.

35. $\lim_{x\to 2^-}\dfrac{x}{x^2 - 4}$.

36. $\lim_{x\to 2}\dfrac{x}{x^2 - 4}$.

37. $\lim_{y\to 6^+}\dfrac{y + 6}{y^2 - 36}$.

38. $\lim_{y\to 6^-}\dfrac{y + 6}{y^2 - 36}$.

39. $\lim_{y\to 6}\dfrac{y + 6}{y^2 - 36}$.

40. $\lim_{x\to 4^+}\dfrac{3 - x}{x^2 - 2x - 8}$.

41. $\lim_{x\to 4^-}\dfrac{3 - x}{x^2 - 2x - 8}$.

42. $\lim_{x\to 4}\dfrac{3 - x}{x^2 - 2x - 8}$.

43. $\lim_{x\to +\infty}\dfrac{7 - 6x^5}{x + 3}$.

44. $\lim_{t\to -\infty}\dfrac{5 - 2t^3}{t^2 + 1}$.

45. $\lim_{t\to +\infty}\dfrac{6 - t^3}{7t^3 + 3}$.

46. $\lim_{x\to 0^+}\dfrac{x}{|x|}$.

47. $\lim_{x\to 0^-}\dfrac{x}{|x|}$.

48. $\lim_{x\to 3^-}\dfrac{1}{|x - 3|}$.

49. $\lim_{x\to 9}\dfrac{x - 9}{\sqrt{x} - 3}$.

50. $\lim_{y\to 4}\dfrac{4 - y}{2 - \sqrt{y}}$.

51. Let $f(x) = \begin{cases} x - 1, & x \le 3 \\ 3x - 7, & x > 3 \end{cases}$

Find:
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$ (c) $\lim_{x\to 3} f(x)$.

52. Let $g(t) = \begin{cases} t^2, & t \ge 0 \\ t - 2, & t < 0 \end{cases}$

Find:
(a) $\lim_{t\to 0^-} g(t)$ (b) $\lim_{t\to 0^+} g(t)$ (c) $\lim_{t\to 0} g(t)$.

53. Find $\lim_{x\to 3} h(x)$ given that

$$h(x) = \begin{cases} x^2 - 2x + 1, & x \ne 3 \\ 7, & x = 3 \end{cases}$$

54. Let $F(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3}, & x \ne -3 \\ k, & x = -3 \end{cases}$

(a) Find k so that $F(-3) = \lim_{x\to -3} F(x)$.

(b) With k assigned the value $\lim_{x\to -3} F(x)$, show that $F(x)$ can be expressed as a polynomial.

2.7 CONTINUITY

A moving physical object cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of a moving object as a single, unbroken curve without gaps, jumps, or holes. Such curves can be described as "continuous." In this section we shall express this intuitive idea mathematically and develop some properties of continuous curves.

Before we give any formal definitions, let us consider some of the ways in which curves can be "discontinuous." In Figure 2.7.1 we have graphed some curves which, because of their behavior at the point c, are not continuous. The curve in Figure 2.7.1*a* has a hole at the point c because the function f is undefined there. For the curves in Figures 2.7.1*b* and 2.7.1*c*, the function f is defined at c, but

$$\lim_{x \to c} f(x) \tag{1}$$

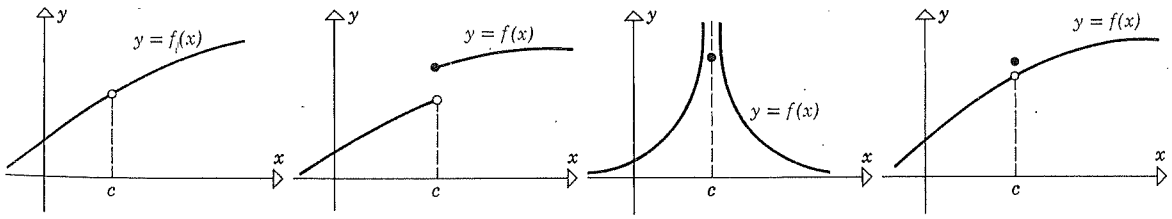


Figure 2.7.1

does not exist, thereby causing a break in the graph. For the curve in Figure 2.7.1*d*, the function f is defined at c, and the limit in (1) exists, yet the graph still has a break at the point c because

$$\lim_{x \to c} f(x) \neq f(c)$$

Based on this discussion, we see that there is a break or discontinuity in the graph of $y = f(x)$ at a point $x = c$ if any of the following conditions occur:

- The function f is undefined at c.
- The limit $\lim_{x \to c} f(x)$ does not exist.
- The function f is defined at c and the limit $\lim_{x \to c} f(x)$ exists, but the value of the function at c and the value of the limit at c are different.

This suggests the following definition.

2.7.1 DEFINITION. A function f is said to be ***continuous at a point c*** if the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

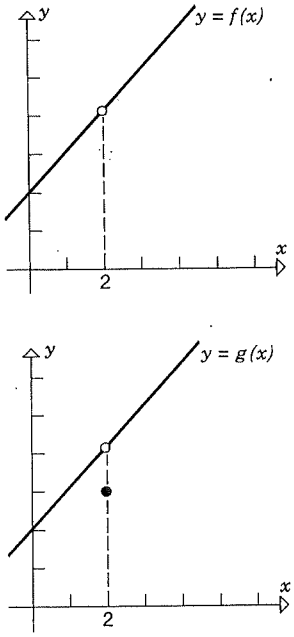


Figure 2.7.2

If one or more of the conditions in this definition fails to hold, then f is called ***discontinuous at c*** and c is called a ***point of discontinuity*** of f. If f is continuous at all points of an open interval (a, b), then f is said to be ***continuous on*** (a, b). A function that is continuous on $(-\infty, +\infty)$ is said to be ***continuous everywhere*** or simply ***continuous.***

Example 1 Let

$$f(x) = \frac{x^2 - 4}{x - 2} \quad \text{and} \quad g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$$

Both f and g are discontinuous at 2 (Figure 2.7.2), the function f because $f(2)$ is undefined, and the function g because $g(2) = 3$, and

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

so that

$$\lim_{x \to 2} g(x) \neq g(2) \qquad \blacktriangleleft$$

REMARK. Some authors define a function to be continuous at c if condition 3 in Definition 2.7.1 holds. This is really equivalent to our definition because if condition 3 holds, then 1 and 2 hold automatically. (Why?) We have stated the three conditions for clarity. However, when we want to show that a function is continuous at a point, we shall just show that condition 3 holds.

Example 2 Show that $f(x) = x^2 - 2x + 1$ is a continuous function.

Solution. We must show that the third condition in Definition 2.7.1 holds for all real c. But

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 - 2x + 1) = c^2 - 2c + 1 = f(c)$$

which shows that the third condition holds. ◀

The foregoing example is a special case of the following general result.

CONTINUITY OF POLYNOMIALS

2.7.2 THEOREM. *Polynomials are continuous functions.*

Proof. If p is a polynomial and c is any real number, then by Example 6 in Section 2.5

$$\lim_{x \to c} p(x) = p(c)$$

which proves the continuity of p at c. Since c is an arbitrary real number, p is continuous everywhere. ∎

Example 3 Show that $f(x) = |x|$ is a continuous function.

Solution. We can write $f(x)$ as

$$f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows from Theorem 2.7.2 that $f(x) = |x|$ is continuous if $x > 0$ or $x < 0$ because $|x|$ is identical to the polynomial x in the former case and identical to the polynomial $-x$ in the latter. Thus, $x = 0$ is the only point in question. At this point we have $f(0) = |0| = 0$, so it remains to show that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 \tag{2}$$

Because the formula for f changes at 0, it will be helpful to consider the one-sided limits at 0 rather than the two-sided limit. We obtain

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Thus, (2) holds and $|x|$ is continuous at $x = 0$. ◀

The following basic result is an immediate consequence of Theorem 2.5.1.

2.7.3 THEOREM. *If the functions f and g are continuous at c, then*

(*a*) $f + g$ *is continuous at* c
(*b*) $f - g$ *is continuous at* c
(*c*) $f \cdot g$ *is continuous at* c
(*d*) f/g *is continuous at* c *if* $g(c) \neq 0$ *and is discontinuous at* c *if* $g(c) = 0$.

We shall prove part (*d*) and leave the remaining proofs as exercises.

Proof of (*d*). If $g(c) = 0$, then f/g is discontinuous at c because $f(c)/g(c)$ is undefined.

Assume $g(c) \neq 0$. We must show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \tag{3}$$

Since f and g are continuous at c,

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c)$$

Thus, by Theorem 2.5.1(*d*)

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)} = \frac{f(c)}{g(c)}$$

which proves (3). ∎

CONTINUITY OF RATIONAL FUNCTIONS

Example 4 Where is

$$h(x) = \frac{x^2 - 9}{x^2 - 5x + 6}$$

continuous?

Solution. The numerator and denominator of h are polynomials and therefore are continuous everywhere by Theorem 2.7.2. Thus, by Theorem 2.7.3(*d*), the ratio is continuous everywhere except at the points where the denominator is zero. Since the solutions of

$$x^2 - 5x + 6 = 0$$

are $x = 2$ and $x = 3$, $h(x)$ is continuous everywhere except at these points. ◀

The result in the foregoing example is a special case of the following general theorem whose proof is left as an exercise.

2.7.4 THEOREM. *A rational function is continuous everywhere except at the points where the denominator is zero.*

CONTINUITY OF COMPOSITIONS

The next theorem is useful for calculating limits of compositions of functions.

2.7.5 THEOREM. *Let* lim *stand for one of the limits* $\lim_{x\to c}$, $\lim_{x\to c^-}$, $\lim_{x\to c^+}$, $\lim_{x\to+\infty}$, *or* $\lim_{x\to-\infty}$. *If* $\lim g(x) = L$ *and if the function* f *is continuous at* L, *then* $\lim f(g(x)) = f(L)$. *That is,* $\lim f(g(x)) = f(\lim g(x))$.

For those who have read Section 2.6, the optional proof of this result appears in Appendix 2.

REMARK. In words, this theorem states that the limit symbol can be moved through a function sign provided the limit of the expression inside the function sign exists and the function is continuous at this limit.

Example 5 We saw in Example 3 that $|x|$ is continuous everywhere, so Theorem 2.7.5 implies that

$$\lim |g(x)| = |\lim g(x)|$$

if $\lim g(x)$ exists. For example,

$$\lim_{x\to 3} |5 - x^2| = |\lim_{x\to 3} (5 - x^2)| = |-4| = 4 \quad \blacktriangleleft$$

The following consequence of Theorem 2.7.5 tells us that compositions of continuous functions are themselves continuous.

2.7.6 THEOREM. *If the function* g *is continuous at the point* c *and the function* f *is continuous at the point* $g(c)$, *then the composition* $f \circ g$ *is continuous at* c.

Proof. We must show that $f \circ g$ satisfies the third condition of Definition 2.7.1 at c. But this is so since we can write

$$\lim_{x\to c} (f \circ g)(x) = \lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(g(c)) = (f \circ g)(c)$$

Theorem 2.7.5 — g is continuous at c ∎

Example 6 The function $|5 - x^2|$ is continuous because it is the composition of $|x|$ with $5 - x^2$, both of which are continuous. ◀

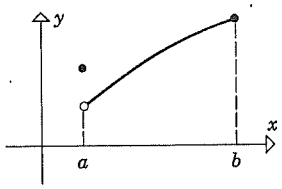
Figure 2.7.3 shows the graphs of three functions defined only on a closed interval $[a, b]$. Obviously the function shown in Figure 2.7.3*a* should be regarded as discontinuous at the left-hand endpoint a, the function in Figure 2.7.3*b* should be regarded as discontinuous at the right-hand endpoint b, and the function in Figure 2.7.3*c* should be regarded as continuous at both endpoints. However, the definition of continuity (Definition 2.7.1) does not apply at the endpoints, since the two-sided limits appearing in parts 2 and 3 of the definition make no sense. At the left-hand endpoint the only sensible limit is the one-sided limit

$$\lim_{x \to a^+} f(x)$$

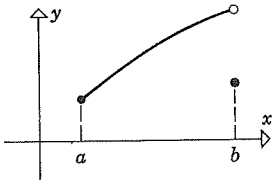
and at the right-hand endpoint the only sensible limit is the one-sided limit

$$\lim_{x \to b^-} f(x)$$

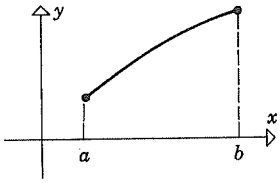
This motivates the following definitions.



(*a*)



(*b*)



(*c*)

Figure 2.7.3

2.7.7 DEFINITION. A function f is called ***continuous from the left at the point c*** if the conditions in the left column below are satisfied, and is called ***continuous from the right at the point c*** if the conditions in the right column are satisfied.

1. $f(c)$ is defined.	1′. $f(c)$ is defined.
2. $\lim_{x \to c^-} f(x)$ exists.	2′. $\lim_{x \to c^+} f(x)$ exists.
3. $\lim_{x \to c^-} f(x) = f(c)$.	3′. $\lim_{x \to c^+} f(x) = f(c)$.

2.7.8 DEFINITION. A function f is said to be ***continuous on a closed interval*** $[a, b]$ if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

The reader should have no trouble deducing the appropriate definitions of continuity on intervals of the form $[a, +\infty)$, $(-\infty, b]$, $[a, b)$, and $(a, b]$.

Example 7 If f denotes the function graphed in Figure 2.7.3*a*, then

$$f(a) \neq \lim_{x \to a^+} f(x) \quad \text{and} \quad f(b) = \lim_{x \to b^-} f(x)$$

Thus, f is continuous from the left at b, but is not continuous from the right at a. ◀

Example 8 Show that $f(x) = \sqrt{9 - x^2}$ is continuous on the closed interval $[-3, 3]$.

Solution. Observe that the domain of f is the interval $[-3, 3]$. For c in the interval $(-3, 3)$ we have by Theorem 2.5.1(*e*)

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

so that f is continuous on $(-3, 3)$. Also,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = 0 = f(3)$$

and

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = 0 = f(-3)$$

so that f is continuous on $[-3, 3]$. ◀

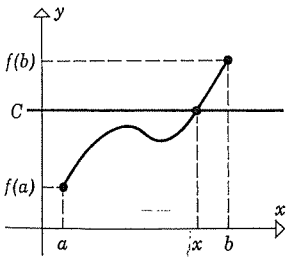


Figure 2.7.4

INTERMEDIATE VALUE THEOREM

If f is continuous on a closed interval $[a, b]$, and we draw a horizontal line crossing the y-axis between the numbers $f(a)$ and $f(b)$ (Figure 2.7.4), then it is geometrically obvious that this line will cross the curve $y = f(x)$ at least once over the interval $[a, b]$. Stated another way, a continuous function must assume every possible value between $f(a)$ and $f(b)$ as x varies from a to b. This idea is formalized in the following theorem.

Intermediate Value Theorem

2.7.9 THEOREM. *If f is continuous on a closed interval $[a, b]$ and C is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = C$.*

(See Figure 2.7.4.) This theorem, though intuitively obvious, is not easy to prove. The proof can be found in most advanced calculus texts.

The following useful result is an immediate consequence of the Intermediate Value Theorem.

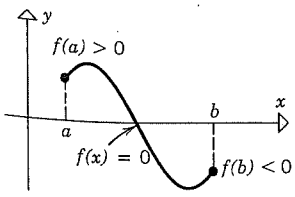


Figure 2.7.5

2.7.10 THEOREM. *If f is continuous on $[a, b]$, and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).*

The proof is left as an exercise, but the result is illustrated in Figure 2.7.5 in the case where $f(a) > 0$ and $f(b) < 0$.

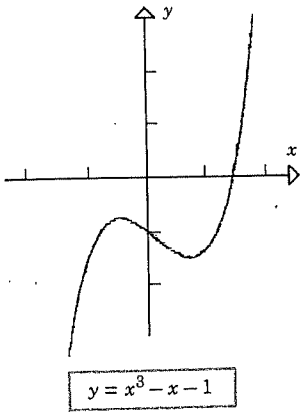


Figure 2.7.6

Theorem 2.7.10 is the basis for various numerical techniques for approximating solutions of equations of the form $f(x) = 0$.

Example 9 The equation

$$x^3 - x - 1 = 0$$

cannot be solved readily by factoring because the left side has no simple factors. However, from Figure 2.7.6, which was generated on a microcomputer, we see that there is a solution in the interval $(1, 2)$. This can also be seen from Theorem 2.7.10 by letting $f(x) = x^3 - x - 1$, and noting that $f(1) = -1$ and $f(2) = 5$ have opposite signs.

To pinpoint the solution more exactly, we can divide the interval $(1, 2)$ into 10 equal parts and evaluate $f(x)$ at each point of subdivision using a calculator (Table 2.7.1). From the table we see that $f(1.3)$ and $f(1.4)$ have opposite signs, so there is a solution in the subinterval $(1.3, 1.4)$. If we use the midpoint 1.35 as an approximation to the solution, then our error is at most half the subinterval length or .05. If desired, we could reduce the error to at most .005 by subdividing the subinterval $(1.3, 1.4)$ into 10 parts and repeating the sign analysis process. With sufficiently many subdivisions, the solution can be determined to any degree of accuracy. ◀

Table 2.7.1

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$f(x)$	-1	$-.77$	$-.47$	$-.10$	.34	.88	1.5	2.2	3.0	3.9	5

We conclude this section by stating an alternative form of the continuity definition that will be useful in later sections.

If we let $h = x - c$ in Definition 2.7.1, then the condition that $x \to c$ is equivalent to $h \to 0$. Thus, Definition 2.7.1 can be restated as follows.

Alternative Form of the Continuity Definition

2.7.11 DEFINITION. A function f is said to be ***continuous at a point c*** if the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{h \to 0} f(c + h)$ exists.
3. $\lim_{h \to 0} f(c + h) = f(c)$.

REMARK. As with Definition 2.7.1, conditions 1 and 2 hold automatically if condition 3 holds. Thus, to prove that a function is continuous at a point, it is only necessary to show that condition 3 holds.

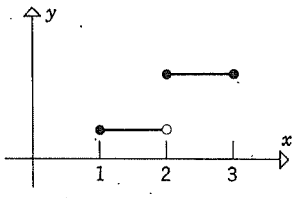
▶ Exercise Set 2.7

In Exercises 1–4, let f be the function whose graph is shown. On which of the following intervals, if any, is f continuous?

(a) $[1, 3]$ (b) $(1, 3)$
(c) $[1, 2]$ (d) $(1, 2)$
(e) $[2, 3]$ (f) $(2, 3)$.

On those intervals where f is discontinuous, state where the discontinuities occur.

1.



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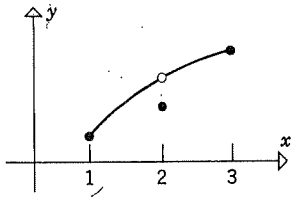

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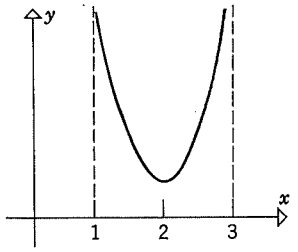

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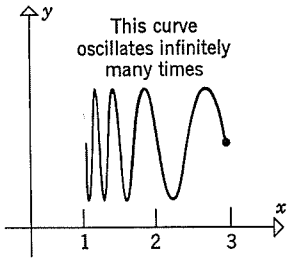

In Exercises 5–16, find the points of discontinuity, if any.

5. $f(x) = x^3 - 2x + 3$.

6. $f(x) = (x - 5)^{17}$.

7. $f(x) = \dfrac{x}{x^2 + 1}$.

8. $f(x) = \dfrac{x}{x^2 - 1}$.

9. $f(x) = \dfrac{x - 4}{x^2 - 16}$.

10. $f(x) = \dfrac{3x + 1}{x^2 + 7x - 2}$.

11. $f(x) = \dfrac{x}{|x| - 3}$.

12. $f(x) = \dfrac{5}{x} + \dfrac{2x}{x + 4}$.

13. $f(x) = |x^3 - 2x^2|$.

14. $f(x) = \dfrac{x + 3}{|x^2 + 3x|}$.

15. $f(x) = \begin{cases} 2x + 3, & x \le 4 \\ 7 + \dfrac{16}{x}, & x > 4. \end{cases}$

16. $f(x) = \begin{cases} \dfrac{3}{x - 1}, & x \ne 1 \\ 3, & x = 1. \end{cases}$

17. Find a value for the constant k that will make the function continuous.

(a) $f(x) = \begin{cases} 7x - 2, & x \le 1 \\ kx^2, & x > 1 \end{cases}$

(b) $f(x) = \begin{cases} kx^2, & x \le 2 \\ 2x + k, & x > 2. \end{cases}$

18. On which of the following intervals is

$$f(x) = \frac{1}{\sqrt{x - 2}}$$

continuous?

(a) $[2, +\infty)$ (b) $(-\infty, +\infty)$
(c) $(2, +\infty)$ (d) $[1, 2)$.

19. (a) Prove that $f(x) = \sqrt{x}$ is continuous on $[0, +\infty)$.

(b) Prove that if $g(x)$ is continuous and nonnegative, then $\sqrt{g(x)}$ is continuous.

20. Prove that if $g(x)$ is continuous, then $|g(x)|$ is continuous.

21. Find all points of discontinuity of the greatest integer function, $f(x) = [x]$. (See Exercise 38, Section 2.3, for the definition of $[x]$.)

22. A function $f(x)$ has a ***removable discontinuity*** at $x = c$ if $\lim_{x \to c} f(x)$ exists, but $f(c)$ either is not defined or $f(c) \neq \lim_{x \to c} f(x)$. If $f(x)$ is redefined to have the value $\lim_{x \to c} f(x)$ at $x = c$, then $f(x)$ will be continuous at $x = c$.

Find all points of discontinuity of $f(x)$, and for each such point state if it is removable or not.

(a) $f(x) = \dfrac{|x|}{x}$

(b) $f(x) = \dfrac{x^2 + 3x}{x + 3}$

(c) $f(x) = \dfrac{x - 2}{|x| - 2}$

(d) $f(x) = \dfrac{x^2 - 4}{x^3 - 8}$

(e) $f(x) = \begin{cases} 2x - 3, & x \leq 2 \\ x^2, & x > 2 \end{cases}$

(f) $f(x) = \begin{cases} 3x^2 + 5, & x \neq 1 \\ 6, & x = 1. \end{cases}$

23. Prove:

(a) part (*a*) of Theorem 2.7.3

(b) part (*b*) of Theorem 2.7.3

(c) part (*c*) of Theorem 2.7.3.

24. Prove Theorem 2.7.4.

25. Let f and g be discontinuous at c. Give examples to show:

(a) $f + g$ can be continuous or discontinuous at c

(b) $f \cdot g$ can be continuous or discontinuous at c.

26. **(For students who have read Section 2.6.)** Let f be defined at c. Prove that f is continuous at c if and only if, given $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

27. Use Theorem 2.7.9 to prove Theorem 2.7.10.

In Exercises 28 and 29, show that the equation has at least one solution in the given interval.

28. $x^3 - 4x + 1 = 0$; $[1, 2]$

29. $x^3 + x^2 - 2x = 1$; $[-1, 1]$

30. Figure 2.7.7 shows the graph of $y = x^4 + x - 1$ generated on a microcomputer. Use the method of Example 9 to approximate the real solutions of $x^4 + x - 1 = 0$ with an error of at most .05.

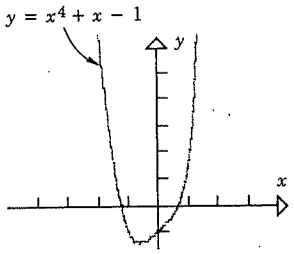

Figure 2.7.7

31. Figure 2.7.8 shows the graph of $y = 5 - x - x^4$ generated on a microcomputer. Use the method of Example 9 to approximate the real solutions of $5 - x - x^4 = 0$ with an error of at most .05.

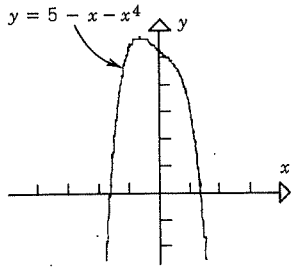

Figure 2.7.8

32. For the equation $x^3 - x - 1 = 0$ discussed in Example 9, show that the real solution is approximately 1.325, with an error of at most .005.

33. Use the fact that $\sqrt{5}$ is a solution of $x^2 - 5 = 0$ to approximate $\sqrt{5}$ with an error of at most

(a) .05 (b) .005.

TUES: 1-6, 10, 11, 12

HUGHES-HALLETT

1.7 INTRODUCTION TO CONTINUITY

This section introduces the idea of *continuity* on an interval and at a point. This leads to the concept of limit, which is investigated in the next section.

Continuity of a Function on an Interval: Graphical Viewpoint

Roughly speaking, a function is said to be *continuous* on an interval if its graph has no breaks, jumps, or holes in that interval. Continuity is important because, as we shall see, continuous functions have many desirable properties.

For example, to locate the zeros of a function, we often look for intervals where the function changes sign. In the case of the function $f(x) = 3x^3 - x^2 + 2x - 1$, for instance, we expect[11] to find a zero between 0 and 1 because $f(0) = -1$ and $f(1) = 3$. (See Figure 1.75.) To be sure that $f(x)$ has a zero there, we need to know that the graph of the function has no breaks or jumps in it. Otherwise the graph could jump across the x-axis, changing sign but not creating a zero. For example, $f(x) = 1/x$ has opposite signs at $x = -1$ and $x = 1$, but no zeros for $-1 \le x \le 1$ because of the break at $x = 0$. (See Figure 1.76.) To be certain that a function has a zero in an interval on which it changes sign, we need to know that the function is defined and continuous in that interval.

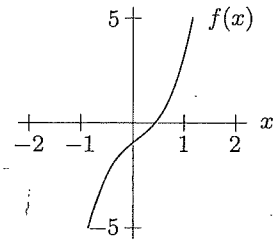

Figure 1.75: The graph of $f(x) = 3x^3 - x^2 + 2x - 1$

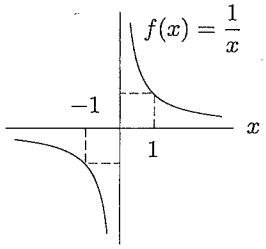

Figure 1.76: No zero although $f(-1)$ and $f(1)$ have opposite signs

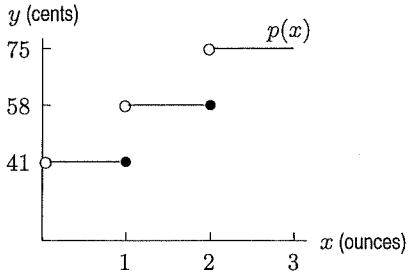

Figure 1.77: Cost of mailing a letter

A continuous function has a graph which can be drawn without lifting the pencil from the paper.

Example: The function $f(x) = 3x^3 - x^2 + 2x - 1$ is continuous on any interval. (See Figure 1.75.)

Example: The function $f(x) = 1/x$ is not defined at $x = 0$. It is continuous on any interval not containing the origin. (See Figure 1.76.)

Example: Suppose $p(x)$ is the price of mailing a first-class letter weighing x ounces. It costs $41¢$ for one ounce or less, $58¢$ between the first and second ounces, and so on. So the graph (in Figure 1.77) is a series of steps. This function is not continuous on any open interval containing a positive integer because the graph jumps at these points.

Which Functions are Continuous?

Requiring a function to be continuous on an interval is not asking very much, as any function whose graph is an unbroken curve over the interval is continuous. For example, exponential functions, polynomials, and the sine and cosine are continuous on every interval. Rational functions are continuous on any interval in which their denominators are not zero. Functions created by adding, multiplying, or composing continuous functions are also continuous.

[11] This is due to the Intermediate Value Theorem, which is discussed on page 48.

The Intermediate Value Theorem

Continuity tells us about the values taken by a function. In particular, a continuous function cannot skip values. For example, the function in the next example must have a zero because its graph cannot skip over the x-axis.

Example 1 What do the values in Table 1.18 tell you about the zeros of $f(x) = \cos x - 2x^2$?

Table 1.18

x	$f(x)$
0	1.00
0.2	0.90
0.4	0.60
0.6	0.11
0.8	−0.58
1.0	−1.46

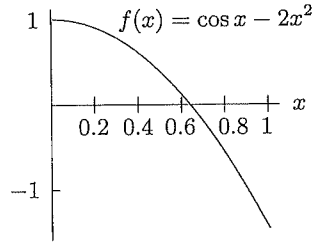

Figure 1.78: Zeros occur where the graph of a continuous function crosses the horizontal axis

Solution Since $f(x)$ is the difference of two continuous functions, it is continuous. We conclude that $f(x)$ has at least one zero in the interval $0.6 < x < 0.8$, since $f(x)$ changes from positive to negative on that interval. The graph of $f(x)$ in Figure 1.78 suggests that there is only one zero in the interval $0 \leq x \leq 1$, but we cannot be sure of this from the graph or the table of values.

In the previous example, we concluded that $f(x) = \cos x - 2x^2$ has a zero between $x = 0$ and $x = 1$ because $f(x)$ is positive at $x = 0$ and negative at $x = 1$. More generally, an intuitive notion of continuity tells us that, as we follow the graph of a continuous function f from some point $(a, f(a))$ to another point $(b, f(b))$, then f takes on all intermediate values between $f(a)$ and $f(b)$. (See Figure 1.79.) This is:

Theorem 1.1: Intermediate Value Theorem

Suppose f is continuous on a closed interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

The Intermediate Value Theorem depends on the formal definition of continuity given in Section 1.8. See also www.wiley.com/college/hugheshallett. The key idea is to find successively smaller subintervals of $[a, b]$ on which f changes from less than k to more than k. These subintervals converge on the number c.

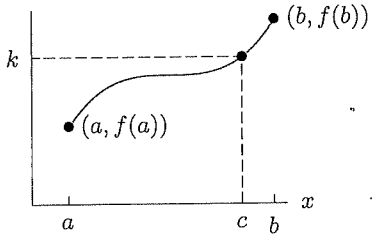

Figure 1.79: The Intermediate Value Theorem

TUES: 1-6, 10, 11, 12

Continuity of a Function at a Point: Numerical Viewpoint

A function is continuous if nearby values of the independent variable give nearby values of the function. In practical work, continuity is important because it means that small errors in the independent variable lead to small errors in the value of the function.

Example: Suppose that $f(x) = x^2$ and that we want to compute $f(\pi)$. Knowing f is continuous tells us that taking $x = 3.14$ should give a good approximation to $f(\pi)$, and that we can get as accurate an approximation to $f(\pi)$ as we want by using enough decimals of π.

Example: If $p(x)$ is the cost of mailing a letter weighing x ounces, then $p(0.99) = p(1) = 41¢$, whereas $p(1.01) = 58¢$, because as soon as we get over 1 ounce, the price jumps up to $58¢$. So a small difference in the weight of a letter can lead to a significant difference in its mailing cost. Hence p is not continuous at $x = 1$.

In other words, if $f(x)$ is continuous at $x = c$, the values of $f(x)$ approach $f(c)$ as x approaches c. In Section 1.8, we discuss the concept of a *limit*, which allows us to define more precisely what it means for the values of $f(x)$ to approach $f(c)$ as x approaches c.

Example 2 Investigate the continuity of $f(x) = x^2$ at $x = 2$.

Solution From Table 1.19, it appears that the values of $f(x) = x^2$ approach $f(2) = 4$ as x approaches 2. Thus f appears to be continuous at $x = 2$. Continuity at a point describes behavior of a function *near* a point, as well as *at* the point.

Table 1.19 *Values of x^2 near $x = 2$*

x	1.9	1.99	1.999	2.001	2.01	2.1
x^2	3.61	3.96	3.996	4.004	4.04	4.41

Exercises and Problems for Section 1.7

Exercises

Are the functions in Exercises 1–10 continuous on the given intervals?

1. $2x + x^{2/3}$ on $[-1, 1]$
2. $2x + x^{-1}$ on $[-1, 1]$
3. $\dfrac{1}{x-2}$ on $[-1, 1]$
4. $\dfrac{1}{x-2}$ on $[0, 3]$
5. $\dfrac{1}{\sqrt{2x-5}}$ on $[3, 4]$
6. $\dfrac{x}{x^2+2}$ on $[-2, 2]$
7. $\dfrac{1}{\cos x}$ on $[0, \pi]$
8. $\dfrac{1}{\sin x}$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$
9. $\dfrac{e^x}{e^x - 1}$ on $[-1, 1]$
10. $\dfrac{e^{\sin\theta}}{\cos\theta}$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$

In Exercises 11–14, show that there is a number c, with $0 \le c \le 1$, such that $f(c) = 0$.

11. $f(x) = x^3 + x^2 - 1$
12. $f(x) = e^x - 3x$
13. $f(x) = x - \cos x$
14. $f(x) = 2^x - 1/x$

15. Are the following functions continuous? Explain.

(a) $f(x) = \begin{cases} x & x \le 1 \\ x^2 & 1 < x \end{cases}$

(b) $g(x) = \begin{cases} x & x \le 3 \\ x^2 & 3 < x \end{cases}$

1-6, 10, 11, 12

1.8 LIMITS

The concept of *limit* is the underpinning of calculus. In Section 1.7, we said that a function f is continuous at $x = c$ if the values of $f(x)$ approach $f(c)$ as x approaches c. In this section, we define a limit, which makes precise what we mean by approaching.

The Idea of a Limit

We first introduce some notation:

> We write $\lim_{x \to c} f(x) = L$ if the values of $f(x)$ approach L as x approaches c.

How should we find L, or even know whether such a number exists? We will look for trends in the values of $f(x)$ as x gets closer to c, but $x \neq c$. A graph from a calculator or computer often helps.

Example 1 Use a graph to estimate $\lim_{\theta \to 0} \left(\dfrac{\sin \theta}{\theta} \right)$. (Use radians.)

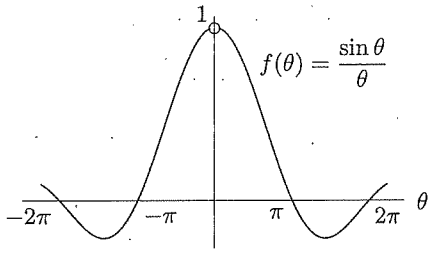

Figure 1.81: Find the limit as $\theta \to 0$

Solution Figure 1.81 shows that as θ approaches 0 from either side, the value of $\sin \theta / \theta$ appears to approach 1, suggesting that $\lim_{\theta \to 0} (\sin \theta / \theta) = 1$. Zooming in on the graph near $\theta = 0$ provides further support for this conclusion. Notice that $\sin \theta / \theta$ is undefined at $\theta = 0$.

Figure 1.81 strongly suggests that $\lim_{\theta \to 0} (\sin \theta / \theta) = 1$, but to be sure we need to be more precise about words like "approach" and "close."

Definition of Limit

By the beginning of the 19th century, calculus had proved its worth, and there was no doubt about the correctness of its answers. However, it was not until the work of the French mathematician Augustin Cauchy (1789–1857) that calculus was put on a rigorous footing. Cauchy gave a formal definition of the limit, similar to the following:

> A function f is defined on an interval around c, except perhaps at the point $x = c$. We define the **limit** of the function $f(x)$ as x approaches c, written $\lim_{x \to c} f(x)$, to be a number L (if one exists) such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c (but $x \neq c$). If L exists, we write
>
> $$\lim_{x \to c} f(x) = L.$$

Shortly, we see how "as close as we want" and "sufficiently close" are expressed using inequalities. First, we look at $\lim_{\theta \to 0} (\sin \theta / \theta)$ more closely (see Example 1).

Example 2 By graphing $y = (\sin\theta)/\theta$ in an appropriate window, find how close θ should be to 0 in order to make $(\sin\theta)/\theta$ within 0.01 of 1.

Solution Since we want $(\sin\theta)/\theta$ to be within 0.01 of 1, we set the y-range on the graphing window to go from 0.99 to 1.01. Our first attempt with $-0.5 \le \theta \le 0.5$ yields the graph in Figure 1.82. Since we want the y-values to stay within the range $0.99 < y < 1.01$, we do not want the graph to leave the window through the top or bottom. By trial and error, we find that changing the θ-range to $-0.2 \le \theta \le 0.2$ gives the graph in Figure 1.83. Thus, the graph suggests that $(\sin\theta)/\theta$ is within 0.01 of 1 whenever θ is within 0.2 of 0. Proving this requires an analytical argument, not just graphs from a calculator.

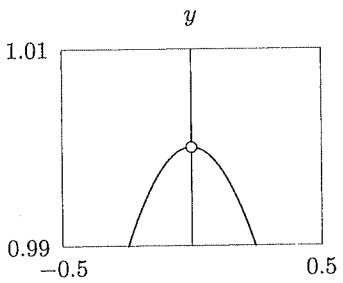

Figure 1.82: $(\sin\theta)/\theta$ with $-0.5 \le \theta \le 0.5$

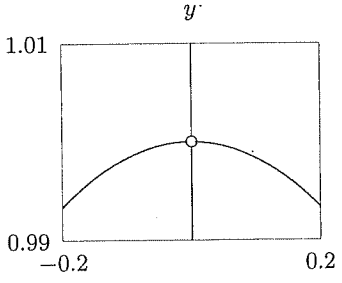

Figure 1.83: $(\sin\theta)/\theta$ with $-0.2 \le \theta \le 0.2$

When we say "$f(x)$ is close to L," we measure closeness by the distance between $f(x)$ and L, expressed using absolute values:

$$|f(x) - L| = \text{Distance between } f(x) \text{ and } L.$$

When we say "as close to L as we want," we use ϵ (the Greek letter epsilon) to specify how close. We write

$$|f(x) - L| < \epsilon$$

to indicate that we want the distance between $f(x)$ and L to be less than ϵ. In Example 2 we used $\epsilon = 0.01$. Similarly, we interpret "x is sufficiently close to c" as specifying a distance between x and c:

$$|x - c| < \delta,$$

where δ (the Greek letter delta) tells us how close x should be to c. In Example 2 we found $\delta = 0.2$.

If $\lim_{x \to c} f(x) = L$, we know that no matter how narrow the horizontal band determined by ϵ in Figure 1.84, there is always a δ which makes the graph stay within that band, for $c - \delta < x < c + \delta$.

Thus we restate the definition of a limit, using symbols:

Definition of Limit

We define $\lim_{x \to c} f(x)$ to be the number L (if one exists) such that for every $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $|x - c| < \delta$ and $x \neq c$, then $|f(x) - L| < \epsilon$.

We have arrived at a formal definition of limit. Let's see if it agrees with our intuition.

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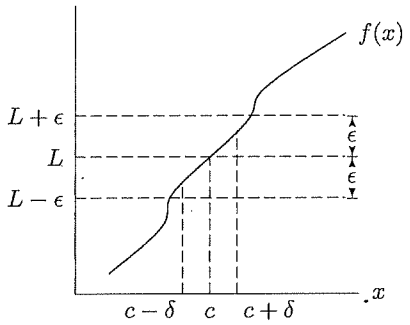

Figure 1.84: What the definition of the limit means graphically

Example 3 Use the definition of limit to show that $\lim_{x \to 3} 2x = 6$.

Solution We must show how, given any $\epsilon > 0$, we can find a $\delta > 0$ such that

$$\text{If } |x - 3| < \delta \text{ and } x \neq 3, \text{ then } |2x - 6| < \epsilon.$$

Since $|2x - 6| = 2|x - 3|$, to get $|2x - 6| < \epsilon$ we require that $|x - 3| < \epsilon/2$. Thus we take $\delta = \epsilon/2$.

It is important to understand that the ϵ, δ definition does not make it easier to calculate limits; rather the ϵ, δ definition makes it possible to put calculus on a rigorous foundation. From this foundation, we can prove the following properties. See Problems 58–60.

Theorem 1.2: Properties of Limits

Assuming all the limits on the right hand side exist:

1. If b is a constant, then $\lim_{x \to c} (bf(x)) = b \left(\lim_{x \to c} f(x) \right)$.
2. $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.
3. $\lim_{x \to c} (f(x)g(x)) = \left(\lim_{x \to c} f(x) \right) \left(\lim_{x \to c} g(x) \right)$.
4. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$.
5. For any constant k, $\lim_{x \to c} k = k$.
6. $\lim_{x \to c} x = c$.

These properties underlie many limit calculations, though we may not acknowledge them explicitly.

Example 4 Explain how the limit properties are used in the following calculation:

$$\lim_{x \to 3} \frac{x^2 + 5x}{x + 9} = \frac{3^2 + 5 \cdot 3}{3 + 9} = 2.$$

Solution We calculate this limit in stages, using the limit properties to justify each step:

$$\lim_{x\to 3}\frac{x^2+5x}{x+9} = \frac{\lim_{x\to 3}(x^2+5x)}{\lim_{x\to 3}(x+9)} \quad \text{(Property 4, since } \lim_{x\to 3}(x+9)\neq 0\text{)}$$

$$= \frac{\lim_{x\to 3}(x^2)+\lim_{x\to 3}(5x)}{\lim_{x\to 3}x+\lim_{x\to 3}9} \quad \text{(Property 2)}$$

$$= \frac{\left(\lim_{x\to 3}x\right)^2+5\left(\lim_{x\to 3}x\right)}{\lim_{x\to 3}x+\lim_{x\to 3}9} \quad \text{(Properties 1 and 3)}$$

$$= \frac{3^2+5\cdot 3}{3+9} = 2. \quad \text{(Properties 5 and 6)}$$

One- and Two-Sided Limits

When we write

$$\lim_{x\to 2} f(x),$$

we mean the number that $f(x)$ approaches as x approaches 2 *from both sides*. We examine values of $f(x)$ as x approaches 2 through values greater than 2 (such as 2.1, 2.01, 2.003) and values less than 2 (such as 1.9, 1.99, 1.994). If we want x to approach 2 only through values greater than 2, we write

$$\lim_{x\to 2^+} f(x)$$

for the number that $f(x)$ approaches (assuming such a number exists). Similarly,

$$\lim_{x\to 2^-} f(x)$$

denotes the number (if it exists) obtained by letting x approach 2 through values less than 2. We call $\lim_{x\to 2^+} f(x)$ a *right-hand limit* and $\lim_{x\to 2^-} f(x)$ a *left-hand limit*. Problems 22 and 23 ask for formal definitions of left and right-hand limits.

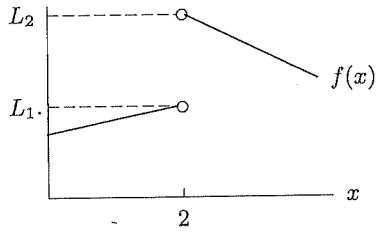

Figure 1.85: Left- and right-hand limits at $x = 2$

For the function graphed in Figure 1.85, we have

$$\lim_{x\to 2^-} f(x) = L_1 \qquad \lim_{x\to 2^+} f(x) = L_2.$$

If the left- and right-hand limits were equal, that is, if $L_1 = L_2$, then it can be proved that $\lim_{x\to 2} f(x)$ exists and $\lim_{x\to 2} f(x) = L_1 = L_2$. Since $L_1 \neq L_2$ in Figure 1.85, we see that $\lim_{x\to 2} f(x)$ does not exist in this case.

When Limits Do Not Exist

Whenever there is no number L such that $\lim_{x\to c} f(x) = L$, we say $\lim_{x\to c} f(x)$ does not exist. Here are three examples in which limits fail to exist.

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Example 5 Explain why $\lim_{x\to 2} \dfrac{|x-2|}{x-2}$ does not exist.

Solution Figure 1.86 shows the problem: The right-hand limit and the left-hand limit are different. For $x > 2$, we have $|x-2| = x-2$, so as x approaches 2 from the right,

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1.$$

Similarly, if $x < 2$, then $|x-2| = 2-x$ so

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{2-x}{x-2} = \lim_{x\to 2^-} (-1) = -1.$$

So if $\lim_{x\to 2} \dfrac{|x-2|}{x-2} = L$ then L would have to be both 1 and -1. Since L cannot have two different values, the limit does not exist.

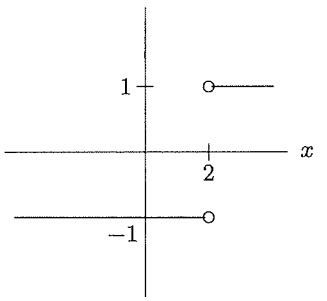

Figure 1.86: Graph of $|x-2|/(x-2)$

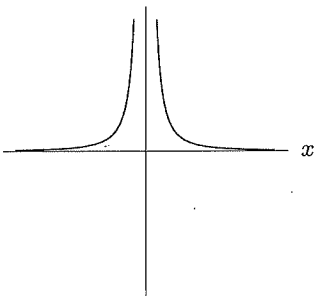

Figure 1.87: Graph of $1/x^2$

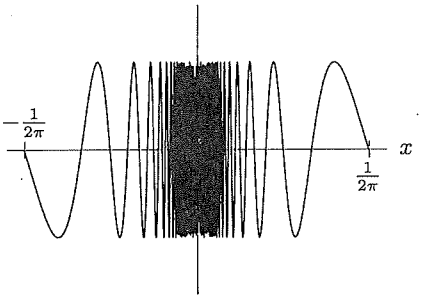

Figure 1.88: Graph of $\sin(1/x)$

Example 6 Explain why $\lim_{x\to 0} \dfrac{1}{x^2}$ does not exist.

Solution As x approaches zero, $1/x^2$ becomes arbitrarily large, so it cannot approach any finite number L. See Figure 1.87. Therefore we say $1/x^2$ has no limit as $x \to 0$.

If $\lim_{x\to a} f(x)$ does not exist because $f(x)$ gets arbitrarily large on both sides of a, we also say $\lim_{x\to a} f(x) = \infty$. So in Example 6 we could say $\lim_{x\to 0} 1/x^2 = \infty$. This behavior may also be described as "diverging to infinity."

Example 7 Explain why $\lim_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ does not exist.

Solution The sine function has values between -1 and 1. The graph in Figure 1.88 oscillates more and more rapidly as $x \to 0$. There are x-values approaching 0 where $\sin(1/x) = -1$. There are also x-values approaching 0 where $\sin(1/x) = 1$. So if the limit existed, it would have to be both -1 and 1. Thus, the limit does not exist.

Limits at Infinity

Sometimes we want to know what happens to $f(x)$ as x gets large, that is, the end behavior of f.

If $f(x)$ gets as close to a number L as we please when x gets sufficiently large, then we write

$$\lim_{x\to\infty} f(x) = L.$$

Similarly, if $f(x)$ approaches L when x is negative and has a sufficiently large absolute value, then we write

$$\lim_{x\to-\infty} f(x) = L.$$

The symbol ∞ does not represent a number. Writing $x \to \infty$ means that we consider arbitrarily large values of x. If the limit of $f(x)$ as $x \to \infty$ or $x \to -\infty$ is L, we say that the graph of f has $y = L$ as a *horizontal asymptote*. Problem 24 asks for a formal definition of $\lim_{x\to\infty} f(x)$.

Example 8 Investigate $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

Solution A graph of $f(x) = 1/x$ in a large window shows $1/x$ approaching zero as x increases in either the positive or the negative direction (See Figure 1.89). This is as we would expect, since dividing 1 by larger and larger numbers yields answers which are smaller and smaller. This suggests that

$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0,$$

and that $f(x) = 1/x$ has $y = 0$ as a horizontal asymptote as $x \to \pm\infty$.

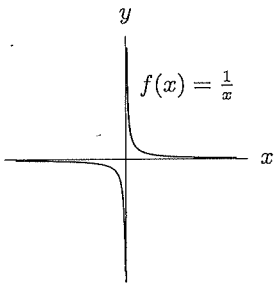

Figure 1.89: The end behavior of $f(x) = 1/x$

Definition of Continuity

We can now give a precise definition of continuity using limits.

The function f is **continuous** at $x = c$ if f is defined at $x = c$ and if

$$\lim_{x\to c} f(x) = f(c).$$

In other words, $f(x)$ is as close as we want to $f(c)$ provided x is close enough to c. The function is **continuous on an interval** $[a, b]$ if it is continuous at every point in the interval.[12]

Constant functions and $f(x) = x$ are continuous for all x. Using the continuity of sums and products, we can show that any polynomial is continuous. Proving that $\sin x$, $\cos x$, and e^x are

[12] If c is an endpoint of the interval, we define continuity at $x = c$ using one-sided limits at c.

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continuous is more difficult. The following theorem, based on the properties of limits on page 53, makes it easier to decide whether certain combinations of functions are continuous.

Theorem 1.3: Continuity of Sums, Products, and Quotients of Functions

Suppose that f and g are continuous on an interval and that b is a constant. Then, on that same interval,

1. $bf(x)$ is continuous.
2. $f(x) + g(x)$ is continuous.
3. $f(x)g(x)$ is continuous.
4. $f(x)/g(x)$ is continuous, provided $g(x) \neq 0$ on the interval.

We prove the third of these properties.

Proof Let c be any point in the interval. We must show that $\lim_{x \to c}(f(x)g(x)) = f(c)g(c)$. Since $f(x)$ and $g(x)$ are continuous, we know that $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$. So, by the first property of limits in Theorem 1.2,

$$\lim_{x \to c}(f(x)g(x)) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right) = f(c)g(c).$$

Since c was chosen arbitrarily, we have shown that $f(x)g(x)$ is continuous at every point in the interval.

Theorem 1.4: Continuity of Composite Functions

If f and g are continuous, and if the composite function $f(g(x))$ is defined on an interval, then $f(g(x))$ is continuous on that interval.

Assuming the continuity of $\sin x$ and e^x, Theorem 1.4 shows us, for example, that $\sin(e^x)$ and $e^{\sin x}$ are both continuous.

Although we now have a formal definition of continuity, some properties of continuous functions, such as the Intermediate Value Theorem, can be difficult to prove. For a further treatment of limits and continuity, see www.wiley.com/college/hugheshallett.

Exercises and Problems for Section 1.8

Exercises

1. Use Figure 1.90 to give approximate values for the following limits (if they exist).

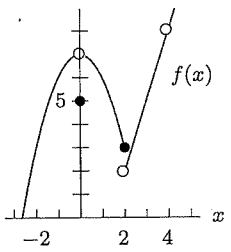

Figure 1.90

(a) $\lim_{x \to -2} f(x)$ **(b)** $\lim_{x \to 0} f(x)$

(c) $\lim_{x \to 2} f(x)$ **(d)** $\lim_{x \to 4} f(x)$

2. Use Figure 1.91 to estimate the limits if they exist:

(a) $\lim_{x\to 0} f(x)$ **(b)** $\lim_{x\to 1} f(x)$

(c) $\lim_{x\to 2} f(x)$ **(d)** $\lim_{x\to 3^-} f(x)$

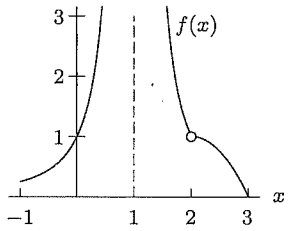

Figure 1.91

3. Using Figures 1.92 and 1.93, estimate

(a) $\lim_{x\to 1^-} (f(x)+g(x))$ **(b)** $\lim_{x\to 1^+} (f(x)+2g(x))$

(c) $\lim_{x\to 1^-} f(x)g(x)$ **(d)** $\lim_{x\to 1^+} \dfrac{f(x)}{g(x)}$

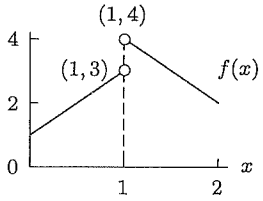

Figure 1.92

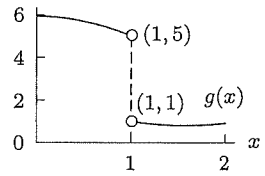

Figure 1.93

Estimate the limits in Exercises 4–5 graphically.

4. $\lim_{x\to 0} \dfrac{|x|}{x}$ **5.** $\lim_{x\to 0} x \ln |x|$

6. Does $f(x) = \dfrac{|x|}{x}$ have right or left limits at 0? Is $f(x)$ continuous?

Use a graph to estimate each of the limits in Exercises 7–10. Use radians unless degrees are indicated by $\theta°$.

7. $\lim_{\theta\to 0} \dfrac{\sin(2\theta)}{\theta}$ **8.** $\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta}$

9. $\lim_{\theta\to 0} \dfrac{\sin\theta°}{\theta°}$ **10.** $\lim_{\theta\to 0} \dfrac{\theta}{\tan(3\theta)}$

For the functions in Exercises 11–13, use algebra to evaluate the limits $\lim_{x\to a+} f(x)$, $\lim_{x\to a-} f(x)$, and $\lim_{x\to a} f(x)$ if they exist. Sketch a graph to confirm your answers.

11. $a = 4, \quad f(x) = \dfrac{|x-4|}{x-4}$

12. $a = 2, \quad f(x) = \dfrac{|x-2|}{x}$

13. $a = 3, \quad f(x) = \begin{cases} x^2 - 2, & 0 < x < 3 \\ 2, & x = 3 \\ 2x + 1, & 3 < x \end{cases}$

14. Estimate how close θ should be to 0 to make $(\sin\theta)/\theta$ stay within 0.001 of 1.

15. Write the definition of the following statement both in words and in symbols:

$$\lim_{h\to a} g(h) = K.$$

Problems

In Problems 16–19, is the function continuous for all x? If not, say where it is not continuous and explain in what way the definition of continuity is not satisfied.

16. $f(x) = 1/x$

17. $f(x) = \begin{cases} |x|/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

18. $f(x) = \begin{cases} x/x & x \neq 0 \\ 1 & x = 0 \end{cases}$

19. $f(x) = \begin{cases} 2x/x & x \neq 0 \\ 3 & x = 0 \end{cases}$

20. By graphing $y = (1+x)^{1/x}$, estimate $\lim_{x\to 0} (1+x)^{1/x}$. You should recognize the answer you get. What does the limit appear to be?

21. What does a calculator suggest about $\lim_{x\to 0+} xe^{1/x}$? Does the limit appear to exist? Explain.

In Problems 22–24, modify the definition of limit on page 52 to give a definition of each of the following.

22. A right-hand limit **23.** A left-hand limit

24. $\lim_{x\to\infty} f(x) = L$

25. If $p(x)$ is the function on page 47 giving the price of mailing a first-class letter, explain why $\lim_{x\to 1} p(x)$ does not exist.

26. Investigate $\lim_{h\to 0} (1+h)^{1/h}$ numerically.